GRAPHIC ALGEBRA

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GRAPHIC ALGEBRA

BY

ARTHUR SCHULTZE, PH.D.

ASSISTANT PROFESSOR OF MATHEMATICS, NEW YORK UNIVERSITY
HEAD OF THE DEPARTMENT OF MATHEMATICS, HIGH
SCHOOL OF COMMERCE, NEW YORK

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PREFACE

It is now generally conceded that graphic methods are not only of great importance for practical work and scientific investigation, but also that their educational value for secondary instruction is very considerable. Consequently, an increasing number of schools have introduced graphic algebra into their courses, and several elementary text-books on graphs have been published.

This book gives an elementary presentation of all the fundamental principles included in such courses, and contains in addition a number of methods which are shorter and require less numerical work than those usually given. Thus, for the solution of a cubic or biquadratic by the customary method a great deal of calculation is necessary to determine the coordinates of a number of points. To avoid these calculations and to make the work truly graphic, the author has devised a series of methods for solving quadratics, cubics, and biquadratics by means of a standard curve and straight lines or circles.

Two of these methods — the solution of quadratics by a parabola (§ 30) and of incomplete cubics by a cubic parabola (§ 49) — are but slight modifications of methods previously known; the others are original with the author, who first published them in a paper read before the American Mathematical Society in April, 1905.

The author desires to acknowledge his indebtedness to Mr. Charles E. Deppermann for the careful reading of the proofs and for verifying the results of the examples.

ARTHUR SCHULTZE.

NEW YORK, December, 1907.

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GRAPHIC ALGEBRA

PART I

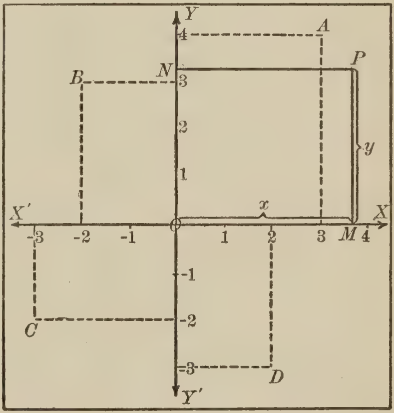
GENERAL GRAPHIC METHODS

CHAPTER I

DEFINITIONS

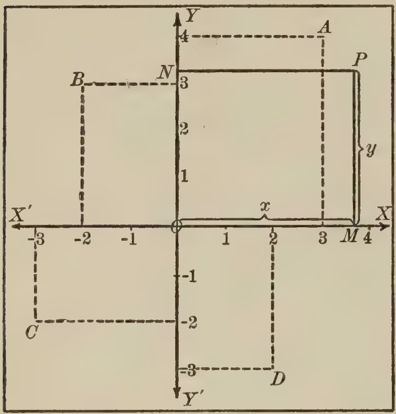
1. Location of a point. If two fixed straight lines XX' and YY' meet in O at right angles, and $PM \perp XX'$, and $PN \perp YY'$, then the position of the point P is determined if the lengths of PM and PN are given.

2. Coördinates. The lines PN and PM are called the **coördinates** of point P; PN, or its equal OM, is the **abscissa**; and PM, or its equal ON, is the **ordinate** of point P. The abscissa is usually denoted by x, the ordinate by y.



The line XX' is called the **x-axis** or the **axis of abscissas**, YY' the **y-axis** or the **axis of ordinates**. The point O is the **origin**, and M and N are the *projections* of P upon the axes. Abscissas measured to the *right of the origin*, and ordinates *above the x-axis*, are considered **positive**; hence coördinates lying in opposite directions are **negative**.

3. The point whose abscissa is x, and whose ordinate is y, is usually denoted by (x, y). Thus the points A, B, C, and D are respectively represented by $(3, 4)$, $(-2, 3)$, $(-3, -2)$, and $(2, -3)$.



The process of locating a point whose coördinates are given is called **plotting the point.**

4. Since there are other methods of determining the location of a point, the coördinates used here are sometimes, for the sake of distinction, called **rectangular coördinates.**

NOTE 1. While usually the same length is taken to represent the unit of the abscissas and the unit of the ordinates, it is sometimes convenient to draw the x and the y on different scales.

NOTE 2. Graphical constructions are greatly facilitated by the use of cross-section paper, *i.e.* paper ruled with two sets of equidistant and parallel lines intersecting at right angles. (See diagram on page 29.)

EXERCISE 1

1. Plot the points: $(3, 2)$, $(4, -1)$, $(-3, 2)$, $(-3, -3)$.

2. Plot the points: $(-2, 3)$, $(-5, 0)$, $(4, -3)$, $(0, 4)$.

3. Plot the points: $(0, -4)$, $(4, 0)$, $(0, 0)$.

4. Draw the triangle whose vertices are respectively $(4, 1)$, $(-1, 3)$, and $(1, -2)$.

5. Plot the points $(-2, 1)$ and $(2, -3)$, and measure their distance.

6. What is the distance of the point $(3, 4)$ from the origin?

7. Where do all points lie whose ordinates equal 4?

8. Where do all points lie whose abscissas equal zero?

9. Where do all points lie whose ordinate equals zero?

10. What is the locus of (x, y) if $y = 3$?

11. If a point lies in the x-axis, which of its coördinates is known?

12. What are the coördinates of the origin?

CHAPTER II

GRAPHIC REPRESENTATION OF A FUNCTION OF ONE VARIABLE

5. Definitions. An expression involving one or several letters is called a **function** of these letters.

$x^2 - x + 7$ is a function of x.

$\sqrt{y} - \frac{3}{y} - y^2$ is a function of y.

$2\,xy - y^2 + 3\,y^3$ is a function of x and y.

If the value of a quantity changes, the value of a function of this quantity will change, *e.g.* if x assumes successively the values 1, 2, 3, 4, $x^2 - x + 7$ will respectively assume the values 7, 9, 13, 19. If x increases gradually from 1 to 2, $x^2 - x + 7$ will change gradually from 7 to 9.

A **variable** is a quantity whose value changes in the same discussion.

A **constant** is a quantity whose value does not change in the same discussion.

In the example of the preceding article, x is supposed to change, hence it is a variable, while 7 is a constant.

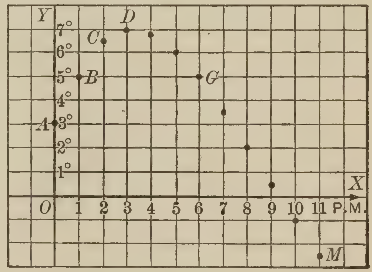
6. Temperature graph. A convenient method for the representation of the various values of a function of a letter, when this letter changes, is the method of representing these values graphically; that is, by a diagram. This method is frequently used to represent in a concise manner a great many data referring to facts taken from physics, chemistry, technology, economics, etc.

To give first an example of one of these applications, let us suppose that we have measured the temperatures at all

hours, from 12 M. to 11 P.M., on a certain day, and that we have found:

At 12 M.	3° C.	At 6 P.M.	5° C.
At 1 P.M.	5° C.	At 7 P.M.	$3\frac{1}{2}$° C.
At 2 P.M.	$6\frac{1}{2}$° C.	At 8 P.M.	2° C.
At 3 P.M.	7° C.	At 9 P.M.	$\frac{1}{2}$° C.
At 4 P.M.	$6\frac{3}{4}$° C.	At 10 P.M.	-1° C.
At 5 P.M.	6° C.	At 11 P.M.	$-2\frac{1}{2}$° C.

To represent graphically one of these facts, *e.g.* the temperature at 6 P.M. was 5°, construct a point G, whose abscissa is 6 and whose ordinate is 5, taking any convenient lengths as units. Representing in a similar way the temperatures at all hours, we obtain the points (0, 3), (1, 5), (2, $6\frac{1}{2}$), (3, 7), etc., *i.e.* $A, B, C, D, \ldots M$.



The diagram thus constructed contains all the information given in the table, but it gives it in a clear and concise form, that at once impresses upon the eye the relative values of the temperatures and their changes.

In a similar manner we may plot the temperatures at any time between 12 M. and 11 P.M. Thus, to represent the fact that the temperature at 1.30 P.M. was 6°, construct the point ($1\frac{1}{2}$, 6).

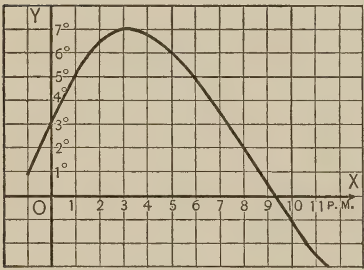
7. If we represented the temperatures of every moment between 12 M. and 11 P.M., we would obtain an uninterrupted sequence of points, or a curved line, as shown in the next diagram. This curve is said to be a **graphical representation** or a **graph** of the temperatures from 12 M. to 11 P.M. It is, of course, not possible to construct an infinite number of points,

hence every graph constructed in the above manner is only an approximation, whose accuracy depends upon the number of points constructed.

To find from the diagram the temperature at any time, *e.g.* at 2.30, measure the ordinate which corresponds to the abscissa $2\frac{1}{2}$; to find when the temperature was 4°, measure the abscissa that corresponds to the ordinate 4, etc.

EXERCISE 2

1. From the diagram find approximate answers to the following questions:



a. Determine the temperature at:

5 P.M., 1.30 P.M., 5.45 P.M., 11.45 A.M.

b. At what hour or hours was the temperature 6°, 5°, 1°, – 1°, 0° ?

c. At what hour was the temperature highest ?

d. What was the highest temperature ?

e. During what hours was the temperature above 5° ?

f. During what hours was the temperature between 3° and 4° ?

g. During what hours was the temperature above 0° ?

h. During what hours was the temperature below 0° ?

i. How much higher was the temperature at 4 than at 8 P.M. ?

k. At what hour was the temperature the same as at 1 P.M. ?

l. During what hours did the temperature increase ?

m. During what hours did the temperature decrease ?

n. Between which two successive hours did the temperature change least ?

o. Between which two successive hours did the temperature increase most rapidly ?

2. Construct a diagram containing the graphs of the mean temperatures of the following four cities:

	JAN. 1	FEB. 1	MARCH 1	APRIL 1	MAY 1	JUNE 1	JULY 1	AUG. 1	SEPT. 1	OCT. 1	NOV. 1	DEC. 1	AVERAGE
New York City	30	32	37	48	60	69	74	72	66	55	43	34	52
San Francisco	50	52	54	55	57	58	58	59	60	59	56	51	56
Tampa	59	66	66	72	76	80	82	81	80	73	65	63	72
Bismarck	4	10	23	42	54	64	70	68	57	44	26	15	40

a. Which of these cities has the most uniform temperature?

b. Which one has the greatest extremes of temperature?

c. When is the mean temperature in San Francisco the same as in New York?

d. When does the mean temperature of New York rise most rapidly?

e. What is the difference between the mean temperatures of New York and Bismarck on Jan. 15?

3. By using the annexed table represent graphically the greatest amount of water vapor which a cubic meter of air can hold at various temperatures.

Degrees of Centigrade	−25	−20	−15	−10	−5	0	5	10	15	20	25	30	35	40
Grams of Vapor	.8	1.0	1.5	2.3	3.4	4.8	6.9	9.3	12.5	17.1	23.0	30	39.3	50.6

a. Represent graphically by a point air of 30° C. which holds 15 grams of water vapor per cubic meter.

b. If such air would cool, represent the change graphically by a line.

c. At what temperature would such air become saturated, *i.e.* contain all the moisture it can hold? *

* This temperature is called the dew-point.

d. If the same air was cooled to 5°, how many grams of moisture would be condensed per cubic meter?

e. How much more moisture per cubic meter can air of the kind mentioned in Ex. *a* hold? *

[For more statistical data suitable for graphic representation see Appendix II.]

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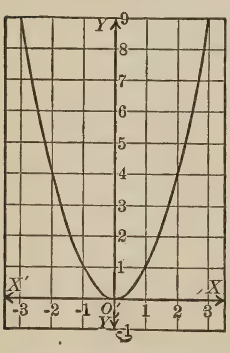
8. Graph of a function. The values of a function for the various values of x may be given in the form of a numerical table. Thus the table on page 84 gives the values of the functions $x^2, x^3, \sqrt{x}, \frac{1}{x},$ for $x = 1, 2, 3, \cdots$ up to 100. The values of functions may, however, be also represented by a graph.

E.g. to construct the graph of x^2 construct a series of points



x	x^2
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

whose abscissas represent x, and whose ordinates are x^2, *i.e.* construct the point $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0) \cdots (3, 9)$, and join the points in order.

If a more exact diagram is required, plot points which lie *between* those drawn above, as $(\frac{1}{2}, \frac{1}{4})$, $(1\frac{1}{2}, 2\frac{1}{4})$, etc.

Since the squares of the numbers increase very rapidly, it is convenient to make the scale unit of the x^2 smaller than that of the x. The graph on page 29 was constructed in this manner.

To find from the graph the square of -2.5, measure the ordinate corresponding to the abscissa -2.5, *i.e.* 6.25. To

* Many meteorological facts can be explained by the graph of Ex. 3, *e.g.* the meaning of "dew-point," relative and absolute humidity, the fact that the mixing of two masses of saturated air of different temperatures produces precipitation, etc.

find $\sqrt{7}$, measure the abscissa whose ordinate is 7, *i.e.* $+2.6$ or -2.6.

Ex. Draw the graph of $\frac{1}{2}x^2 - \frac{1}{5}x - 3$.

To obtain the values of the functions for the various values of x, the following arrangement may be found convenient:

(Compute each column before commencing the next, and see table on page 84.)

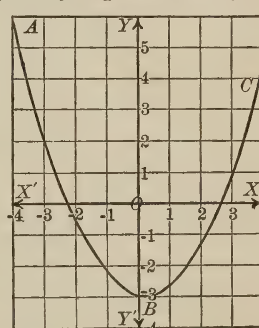
x	x^2	$\frac{1}{2}x^2$	$-\frac{1}{5}x$	$\frac{1}{2}x^2 - \frac{1}{5}x - 3$
-4	16	8	.8	5.8
-3	9	4.5	.6	2.1
-2	4	2	.4	$-.6$
-1	1	.5	.2	-2.3
0	0	0	0	-3
1	1	.5	$-.2$	-2.7
2	4	2	$-.4$	-1.4
3	9	4.5	$-.6$	.9
4	16	8	$-.8$	4.2

Locating the points $(-4, 5.8)$, $(-3, 2.1)$, $(-2, -.6)$, $\cdots$, $(4, 4.2)$, and joining in order produces the graph ABC.

9. For brevity, the function is frequently represented by a single letter, as y. Thus, in the above example,

$$y = \frac{1}{2}x^2 - \frac{1}{5}x - 3;$$

if $x = \frac{1}{2}$, we find from the graph $\frac{1}{2}x^2 - \frac{1}{5}x - 3$ or $y = -3$, if $x = 2\frac{1}{2}$, $y = -.4$, etc.



For values of x greater than 4, the function will obviously be always positive and increase when x increases. Hence the curve will continue to go upward beyond C, and similarly above A.

Graphs should always be drawn until they reach their ultimate direction at both ends.

10. The graph of an equation of the form of $ax^2 + bx + c$ is called a **parabola**.

Thus the graph of $\frac{1}{2}x^2 - \frac{1}{5}x - 3$ is a parabola.

EXERCISE 3

Draw the graphs of the following functions: *

1. $x + 2$.
2. $3x + 5$.
3. $2x - 7$.
4. $\frac{3}{2}x$.
5. $1 - x$.
6. $2 - 3x$.
7. $-3x$.
8. $\frac{1}{4}x^2$.
9. $x^2 - 1$.
10. $x^2 + x$.
11. $x^2 - 2x$.
12. $4x - x^2$.
13. $x^2 - 4x + 4$.
14. $x^2 - x - 5$.
15. $x^2 - 3x - 8$.
16. $x^2 + x - 2$.
17. $x^2 - x + 1$.
18. $6 + x - x^2$.
19. $2 - x - x^2$.
20. $10 - 3x - x^2$.
21. $2x^2 + 5x - 20$.
22. x^3.
23. $x^3 - 2x$.
24. $x^3 - x + 1$.

25. Draw the graph of x^2 from $x = -4$ to $x = 4$, and from the diagram find:

a. $(3.5)^2$; *b.* $(-1.5)^2$; *c.* $(-2.8)^2$; *d.* $(1.9)^2$;
e. $\sqrt{6.25}$; *f.* $\sqrt{12.25}$; *g.* $\sqrt{5}$; *h.* $\sqrt{.3}$.

26. Draw the graph of $x^2 - 4x + 2$ from $x = -1$ to $x = 4$, and from the diagram determine:

(*a*) The values of the function if $x = -\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}$.
(*b*) The values of x, if $x^2 - 4x + 2$ equals $-2, 1, 1\frac{1}{2}$.
(*c*) The smallest value of the function.
(*d*) The value of x that produces the smallest value of the function:
(*e*) The values of x that make $x^2 - 4x + 2 = 0$.
(*f*) The roots of the equation $x^2 - 4x + 2 = 0$.
(*g*) The roots of the equation $x^2 - 4x + 2 = -1$.
(*h*) The roots of the equation $x^2 - 4x + 2 = 2$.

* If necessary, use for the ordinates a smaller unit than for the abscissas.

27. Draw the graph of $y = 2 + 2x - x^2$, from $x = -2$ to $x = 4$, and from the diagram determine:

(*a*) The values of y, *i.e.* the function, if $x = \frac{1}{2}, -1\frac{1}{2}, 2\frac{1}{4}$.
(*b*) The values of x if $y = -2$.
(*c*) The greatest value of the function.
(*d*) The value of x that produces the greatest value of y.
(*e*) The values of x if the function equals zero.
(*f*) The roots of the equation $2 + 2x - x^2 = 0$.
(*g*) The values of x if $y = 1$.
(*h*) The roots of the equation $2 + 2x - x^2 = 1$.

28. The formula for the distance traveled by a falling body is $S = \frac{1}{2}gt^2$.

(*a*) Represent $\frac{1}{2}gt^2$ graphically from $t = 0$ to $t = 5$. (Assume $g = 10$ meters, and make the scale unit of the t equal to 10 times the scale unit of the $\frac{1}{2}gt^2$.)
(*b*) How far does a body fall in $2\frac{1}{2}$ seconds?
(*c*) In how many seconds does a body fall 25 meters?

---

11. **A function of the first degree** is an integral rational function involving only the first power of the variable.

Thus, $4x + 7$ or $ax + b + c$ are functions of the first degree.

12. It can be proved that the **graph of a function of the first degree** is a straight line, hence two points are sufficient for the construction of these graphs. (This is true if the abscissas and ordinates are drawn on different scales or on the same scale.)

It can easily be shown that the preceding proposition is true for any particular example, *e.g.* $3x + 2$.

If $\qquad x = -3, -2, -1, 0, 1, 2, 3,$
then $\qquad 3x + 2 = -7, -4, -1, 2, 5, 8, 11;$

i.e. if x increases by 1, $3x + 2$ increases by 3. Hence if a straight line be drawn through $(-3, -7)$ and $(-2, -4)$, this line will ascend 3 units from $x = -3$ to $x = -2$. Obviously the prolongation of this line will ascend at the same rate throughout, and it will pass through $(-1, -1)$, $(0, 2)$, etc.

Instead of plotting $(-3, -7)$ and $(-2, -4)$, any other two points may be taken. It is advisable not to select two points which lie very closely together.

EXERCISE 4

Draw the graph of

1. $3x - 10$.
2. $5x + 2$.
3. $2x - 7$.
4. $2 - 3x$.
5. $6 + x$.
6. $\frac{2}{3}x - 5$.

7. Degrees of the Fahrenheit scale are expressed in degrees of the Centigrade scale by the formula $C. = \frac{5}{9}(F. - 32)$.

(*a*) Draw the graph of $\frac{5}{9}(F. - 32)$, from $F. = -5$, to $F. = 40$.

(*b*) From the diagram find the number of degrees of Centigrade equal to $-1°$ F., 9° F., 14° F., 32° F.

(*c*) Change to Fahrenheit readings: $-10°$ C., 0° C., 1° C.

8. Show that the graphs of $3x + 2$ and $3x - 1$ are parallel lines.

CHAPTER III

GRAPHIC SOLUTION OF EQUATIONS INVOLVING ONE UNKNOWN QUANTITY

13. Degree of an equation. A rational integral equation which contains the nth power of the unknown quantity, but no higher power, is called an equation of the nth degree.

$x^5 - 5x^3 + 2x^2 - 7 = 0$ is an equation of the fifth degree.
$x^3 - 2x^2 - 5x + 1 = 0$ is an equation of the third degree.

A **quadratic equation** is an equation of the second degree.

A **cubic equation** is an equation of the third degree.

A **biquadratic equation** is an equation of the fourth degree.

$x^3 + 2x + 3 = 0$ is a cubic equation.
$x^4 + 3x^3 + 2x - 7 = 0$ is a biquadratic equation.

14. Solution of equations. Since we can graphically determine the values of x that make a function of x equal to zero, it is evidently possible to find graphically the real roots of an equation.

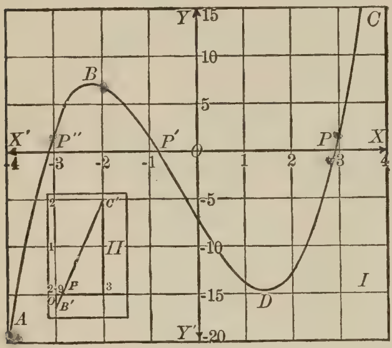
Ex. Find graphically the real roots of the equation

$$x^3 + x^2 - 9x - 7 = 0.$$

(In computing the values of y use table on page 84.)

x	x^2	x^3	$-9x$	x^3+x^2-9x	x^3+x^2-9x-7 or y
-4	16	-64	36	-12	-19
-3	9	-27	27	9	2
-2	4	-8	18	14	7
-1	1	-1	9	9	2
0	0	0	0	0	-7
1	1	1	-9	-7	-14
2	4	8	-18	-6	-13
3	9	27	-27	9	2
4	16	64	-36	44	37

Obviously the values of the function for $x>4$ will increase rapidly, and for values of $x<-4$ will be less than -19. Locating the points $(-4, -19)$, $(-3, 2)$ $(-2, 7)$ . . . $(4, 37)$ and joining produces the graph ABC.



Since ABC intersects the x-axis at three points, P, P', and P'', three values of x make the function zero. Hence there are three roots which, by measuring OP'', OP', and OP, are found to be approximately -3.1, $-.8$, and 2.9.

To find a more exact answer for one of these roots, *e.g.* OP, we draw the portion of the diagram which contains P on a larger scale.

If $x=2.9$, the function equals $-.301$, *i.e.* it is negative. Hence it appears from the diagram that the roots must be larger. Substituting $x=3$ produces $x^3+x^2-9x-7=2$, a positive quantity. The root therefore must lie between 2.9 and 3.

Making the unit of length ten times as large as before, we locate the points $(2.9, -.301)$ and $(3, 2)$, *i.e.* B' and C , in diagram II. Since in nearly all cases small portions of the curve are almost straight lines, we join the two points by a straight line $B'C'$, which intersects the x-axis in P.

The measurement of P gives the root

$$x=2.915.$$

If a greater degree of accuracy is required, a third drawing on a still larger scale must be constructed.

15. The diagram of the last exercise may also be used to find the real roots of an equation of the form $x^3+x^2-9x-7=m$, when m represents a real number.

To solve, *e.g.*, the equation $x^3+x^2-9x-7=2$, determine the points where the function is 2. If cross-section paper is used, the points may be found by inspection, otherwise draw

through $(0, 2)$ a line parallel to the x-axis, and determine the abscissas of the points of intersection with the graph, viz. -3, -1, 3.

16. It can be proved that every equation of the nth degree has n roots; hence if the number of the points of intersection is less than n, the remaining roots are imaginary.

Thus, $x^3 + x^2 - 9x - 7 = 13$ has only one real root, viz. 3.4; hence two roots are imaginary.

If, however, the line parallel to the x-axis is tangent to the curve, the point of tangency represents at least two roots, and hence the preceding paragraph cannot be applied.

EXERCISE 5

Solve graphically the following equations:

1. $4x - 7 = 0$.
2. $2x + 5 = 0$.
3. $6 - x = 0$.
4. $8 - 3x = 0$.
5. $x^2 - x - 6 = 0$.
6. $x^2 - x - 5 = 0$.
7. $x^2 - 2x - 7 = 0$.
8. $x^2 - 6x + 9 = 0$.
9. $x^2 + 5x - 4 = 0$.
10. $x^2 - 5x - 3 = 0$.
11. $x^2 - 3x - 6 = 0$.
12. $x^2 - 2x - 9 = 0$.
13. $3x^2 - 3x - 17 = 0$.
14. $2x^2 - 4x - 15 = 0$.
15. $2x^2 + 10x - 7 = 0$.
16. $3x^2 - 6x - 13 = 0$.
17. $x^3 - 3x - 1 = 0$.
18. $x^3 - 12x + 18 = 0$.
19. $x^3 - 4x + 1 = 0$.
20. $x^3 + x - 3 = 0$.
21. $x^3 + 3x - 11 = 0$.
22. $2x^3 - 6x + 3 = 0$.
23. $x^3 - 5x^2 - 9x + 50 = 0$.
24. $x^3 - 13x^2 + 38x + 17 = 0$.
25. $x^4 - 10x^2 + 8 = 0$.
26. $x^4 - 4x^2 + 4x - 4 = 0$.
27. $x^4 - 6x^3 + 7x^2 + 6x - 7 = 0$.
28. $x^5 - x^4 - 11x^3 + 9x^2 + 18x - 4 = 0$.
29. $2^x + x - 4 = 0$.

30. If $y = x^3 + 5x^2 - 10$,

(*a*) Solve $y = 0$.

(*b*) Solve $y = 5$.

(*c*) Solve $y = -5$.

(*d*) Solve $y = -15$.

(*e*) Determine the number of real roots of the equation $y = -2$.

(*f*) Determine the limits between which m must lie, if $y = m$ has three real roots.

(*g*) Find the value of m that will make two roots equal if $y = m$.

(*h*) Find the greatest value which y may assume for a negative x.

(*i*) Which negative value of x produces the greatest value of y?

31. If $y = x^3 - 7x + 3$,

(*a*) Solve $y = 0$.

(*b*) Solve $y = 3$.

(*c*) Solve $y = -3$.

(*d*) Solve $y = 10$.

(*e*) Solve $y = -15$.

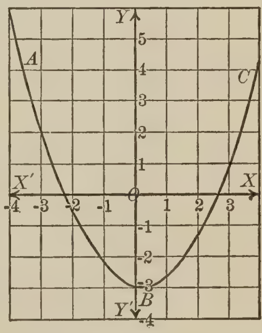
(*f*) Determine the number of real roots if y equals 15, 10, 5, or -7.

(*g*) Determine the number of imaginary roots if $y = -10$, if $y = 12$, if $y = 2$.

CHAPTER IV

GRAPHIC SOLUTION OF EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES

17. Graphs of functions of two unknown quantities. In § 8 the graph of the function $\frac{1}{2}x^2 - \frac{1}{5}x - 3$ was discussed. If $\frac{1}{2}x^2 - \frac{1}{5}x - 3$ is denoted by y, then the ordinate represents the various values of y, and the annexed diagram represents the equation



$$y = \tfrac{1}{2}x^2 - \tfrac{1}{5}x - 3. \qquad (1)$$

The coördinates of every point of the curve satisfy equation (1), and every set of real values of x and y satisfying the equation (1) is represented by the coördinates of a point in the curve.

Similarly, to represent $\dfrac{x^2+x}{y-5} = 2$ graphically solve for y, *i.e.*

$$y = \frac{x^2+x+10}{2},$$

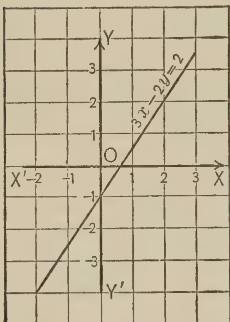
and construct the graph of $\dfrac{x^2+x+10}{2}$.

18. The curve representing an equation is called the **graph** or **locus** of the equation.

19. If an equation containing two unknown quantities can be reduced to the form $y = f(x)$, when $f(x)$ represents a function of x, then the equation can be represented graphically.

Ex. 1. Represent graphically $3x - 2y = 2$.

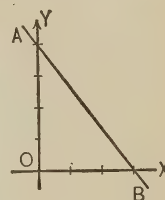
Solving for y, $y = \dfrac{3x-2}{2}$.



Hence, if x equals $-2,\ -1,\ \ 0,\ 1,\ 2,\ 3$;
then y equals $-4,\ -2\frac{1}{2},\ -1,\ \frac{1}{2},\ 2,\ 3\frac{1}{2}$.

Locating the points $(-2, -4)$, $(-1, -2\frac{1}{2})$, etc., and drawing a line through them, we obtain the graph of the equation, which is a straight line.

20. The graph of an equation of the first degree *involving two unknown quantities is always a straight line, and hence it can be constructed if two points are located* (§ 12).

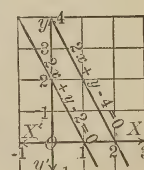


Ex. 2. Draw the locus of $4x + 3y = 12$.

If $x = 0$, $y = 4$; if $y = 0$, $x = 3$.

Hence, locate points (0, 4) and (3, 0), and join them by a straight line *AB*. *AB* is the required graph.

NOTE. Equations of the first degree are called *linear* equations, because their graphs are straight lines.



21. If two linear equations differ only in their absolute terms (*i.e* terms not containing x or y) as $2x + y = 4$ and $2x + y = 2$, their graphs are parallel lines.

EXERCISE 6

Draw the loci of the following equations:

1. $x + y = 4$.
2. $x - 2y = 4$.
3. $2x - 3y = 12$.
4. $x - y = 0$.
5. $x + y = -10$.
6. $y = -4$.
7. $x + y = 0$.
8. $y = 2x$.
9. $12x + 15y = 48$.
10. $x^2 - y + 2 = 0$.
11. $2x^2 - y - x = 0$.
12. $x^3 + y = 0$.
13. $y^2 - x = 2$.
14. $\dfrac{y}{x} - x + 2 + \dfrac{3}{x} = 0$.
15. $x^2 + y^2 = 16$.

16. A body moving with a uniform velocity of 3 yds. per second moves in t seconds a distance $d = 3t$.

Represent this formula graphically.

17. If two variables x and y are directly proportional, then

$$y = cx, \text{ where } c \text{ is a constant.}$$

Show that the graph of two variables that are directly proportional is a straight line passing through the origin (assume for c any convenient number).

18. If two variables x and y are inversely proportional, then

$$y = \frac{c}{x}, \text{ where } c \text{ is a constant.}$$

Draw the locus of this equation if $c = 12$.

19. The temperature remaining the same, the volume v of a gas is inversely proportional to the pressure p. For a certain body of gas, $v = 2$ cubic feet, if $p = 15$ lbs. per square inch. Represent the changes of p and v graphically.

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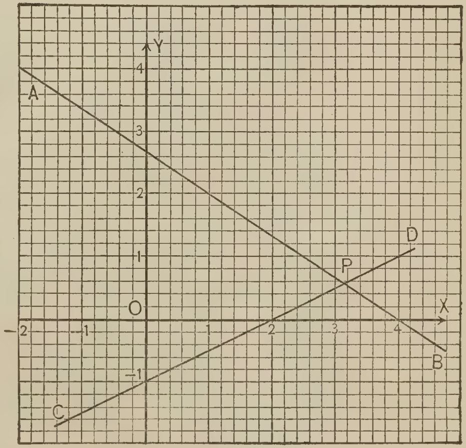
22. Graphical solution of a linear system.

To find the roots of the system:

$$2x + 3y = 8, \quad (1)$$
$$x - 2y = 2. \quad (2)$$



By the method of the preceding article construct the graphs AB and CD of (1) and (2) respectively. The coördinates of every point in AB satisfy the equation (1), but only one point in AB also satisfies equation (2), viz. P, the point of intersection of AB and CD.

By measuring the coördinate of P, we obtain the roots, $x = 3.15$, $y = .57$.

23. The roots of two simultaneous equations are represented by the coördinates of the point (or points) at which their graphs intersect.

24. Since **two straight lines** which are not coincident nor parallel have only one point of intersection, simultaneous linear equations have only one pair of roots.

If two equations are *inconsistent*, as $2x + y - 2 = 0$ and $2x + y - 4 = 0$, their lines are parallel lines (§ 21).

If two equations are *dependent*, their graphs are identical, as

$$\frac{x}{2} + \frac{y}{3} = 1 \text{ and } 3x + 2y = 6.$$

Obviously inconsistent and dependent equations cannot be used to determine the roots of a system of equations.

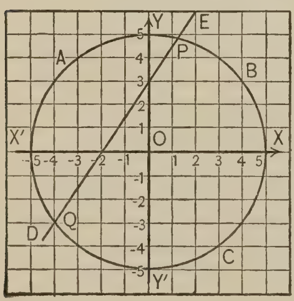
25. Equations of higher degree can have several points of intersection, and hence several pairs of roots.

Ex. 1. Solve graphically the following system:

$$\begin{cases} x^2 + y^2 = 25, & (1) \\ 3x - 2y = -6. & (2) \end{cases}$$

Solving (1) for y, $y = \sqrt{25 - x^2}$.

Therefore, if x equals $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$, y equals respectively $0, \pm 3, \pm 4, \pm 4.5, \pm 4.9, \pm 5, \pm 4.9, \pm 4.5, \pm 4, \pm 3, 0$.



Locating the points $(-5, 0)$, $(-4, +3)$, $(-4, -3)$, etc., and joining, we obtain the graph (a circle) ABC of the equation $x^2 + y^2 = 25$.

Locating two points of equation (2), *e.g.* $(-2, 0)$ and $(0, 3)$, and joining by a straight line, we obtain DE, the graph of $3x - 2y = -6$.

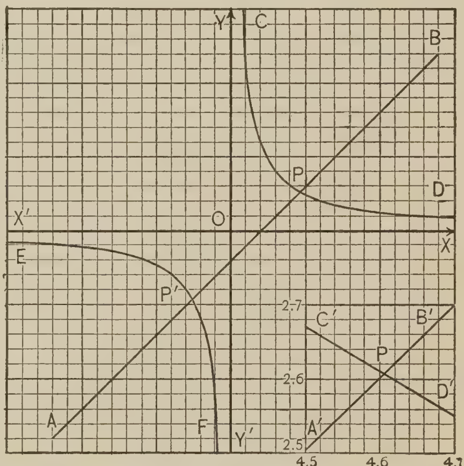
Since the two graphs meet in two points P and Q, there are two pairs of roots, which we find by measurement, $x = 1\frac{1}{3}$, $y = 4\frac{4}{5}$, or $x = -4$, $y = -3$.

Ex. 2. Solve graphically the following system:

$$\begin{cases} xy = 12, & (1) \\ x - y = 2. & (2) \end{cases}$$

From (1) $y = \frac{12}{x}\cdot$ Hence, by substituting for x the values -12, -11, $\cdots$ to $+12$, we obtain the following points: $(-12, -1)$, $(-11, -1\frac{1}{11})$, $(-10, -1\frac{1}{5})$, $(-9, -1\frac{1}{3})$, $(-8, -1\frac{1}{2})$, $(-7, -1\frac{5}{7})$, $(-6, -2)$, $(-5, -2\frac{2}{5})$, $(-4, -3)$, $(-3, -4)$, $(-2, -6)$, $(-1, -12)$, $(0, \pm\infty)$, $(1, 12)$, $(2, 6)$, etc., to $(12, 1)$.

Locating these points and joining them produces the graph of (1), which consists of two separate branches, CD and EF.



Locating two points of equation (2) and joining by a straight line, we have the graph AB of the equation (2).

The coördinates of the two points of intersection P and P' are the required roots. By actual measurement we find $x = 4.5^+$, $y = 2.5^+$, or $x = -2.5$, $y = -4.5$.

To obtain a greater degree of accuracy, the portion of the diagram near P is represented on a larger scale in the small diagram. Since the small part of CD which is represented is almost a straight line, it is sufficient to locate two or three points of this line. By actual measurement we find:

$$x = 4.606,\ y = 2.606.$$

Evidently the second pair is

$$x = -2.606,\ y = -4.606.$$

By increasing the scale further, any degree of accuracy may be obtained.

EXERCISE 7

Solve graphically the following simultaneous equations

1. $\begin{cases} 3x+4y=8, \\ 2x-3y=6. \end{cases}$

2. $\begin{cases} 3x+4y=10, \\ 4x+y=9. \end{cases}$

3. $\begin{cases} 2x-3y=7, \\ 3x+2y=-8. \end{cases}$

4. $\begin{cases} 4x+3y=12, \\ x+5y=6. \end{cases}$

5. $\begin{cases} 3x+5y=7. \\ 5x-y=7. \end{cases}$

6. Show graphically that the following system cannot have finite roots:

$$\begin{cases} 2x-y=2, \\ 2x-y=6. \end{cases}$$

7. Show graphically that the following system is satisfied by an infinite number of roots:

$$\begin{cases} \dfrac{x}{4}+\dfrac{y}{3}=1, \\ 3x+4y=12. \end{cases}$$

8. Without constructing the graphs, determine the relative positions of the loci of $14x-7y+2=0$ and $14x-7y+5=0$.

Solve graphically:

9. $\begin{cases} x^2+y^2=16, \\ x+y=2. \end{cases}$

10. $\begin{cases} x+y=5, \\ xy=6. \end{cases}$

11. $\begin{cases} x-y=1, \\ x^2+y^2=25. \end{cases}$

12. $\begin{cases} x-y=2, \\ xy=8. \end{cases}$

13. $\begin{cases} 4x-5y=10, \\ xy=6. \end{cases}$

14. $\begin{cases} x^2-y^2=4, \\ x=2y. \end{cases}$

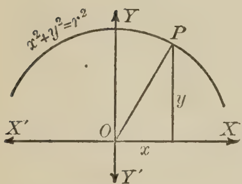
15. $\begin{cases} xy=6, \\ x^2+y^2=25. \end{cases}$

16. $\begin{cases} x^2+xy=12, \\ x^2-y^2=8. \end{cases}$

26. The equation of the circle. *The locus of an equation of the form* $x^2+y^2=r^2$ (1) *is a circle whose center is the origin and whose radius is* r.

For the distance from the origin O of a point P in the locus,

$$OP = \sqrt{x^2 + y^2} \tag{1}$$
$$= \sqrt{r^2} = r.$$



But if the distance of every point in the locus from O is equal to r, then the locus is a circle whose center is O and whose radius is r.

Thus, $x^2 + y^2 = 16$ is a circle whose center is O and whose radius equals 4, $x^2 + y^2 = 10$ is a circle whose center is O and whose radius is $\sqrt{10}$.

Note. The square root of a number can often be represented by the hypotenuse of a right triangle whose arms are rational numbers. Thus, $\sqrt{10} = \sqrt{3^2 + 1^2}$, hence $\sqrt{10}$, equals a line joining (0, 0) and (3, 1).

27. *The locus of expressions of the form*

$$(x - a)^2 + (y - b)^2 = r^2 \tag{2}$$

is a circle whose center is (a, b) *and whose radius equals* r.

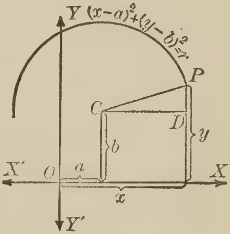
Let P be any point in the locus, and $C = (a, b)$.

Draw $CD \parallel OX$;

$$\overline{CP}^2 = \overline{CD}^2 + \overline{DP}^2.$$

But $CD = x - a$, and $DP = y - b$.

$$\therefore \overline{CP}^2 = (x - a)^2 + (y - b)^2.$$



Hence, from (2), $\overline{CP}^2 = r^2$, or $CP = r$.

I.e. the distance of any point in the locus from C equals r, or the locus is a circle whose center is (a, b) and whose radius is r.

Thus, $(x - 2)^2 + (x + 4)^2 = 8$ represents a circle whose center is $(2, -4)$ and whose radius equals $\sqrt{8}$. Since $\sqrt{8} = \sqrt{2^2 + 2^2}$, it is easily constructed.

Note. The equation $(x - a)^2 + (y - b)^2 = r^2$, however, represents a circle only if the scale units of the abscissas and ordinates are equal. If the two scales are unequal, the locus is an ellipse.

Ex. 1. Construct the locus of

$$x^2+2x+y^2-4y-5=0.$$

Transpose and complete the squares of the expressions involving x and y,

$$(x^2+2x+1)+(y^2-4y+4)=5+5,$$
$$(x+1)^2+(y-2)^2=10.$$

I.e. the required locus is a circle whose center is $(-1, +2)$ and whose radius is $\sqrt{10}$.

Ex. 2. Construct the locus of

$$2x^2+2y^2-3x+6y+3=0.$$

Dividing by 2, and transposing,

$$x^2-\tfrac{3}{2}x+y^2+3y=-\tfrac{3}{2}.$$

Completing the squares,

$$x^2-\tfrac{3}{2}x+(\tfrac{3}{4})^2+y^2+3y+(\tfrac{3}{2})^2=-\tfrac{3}{2}+\tfrac{9}{16}+\tfrac{9}{4},$$
$$(x-\tfrac{3}{4})^2+(y+\tfrac{3}{2})^2=\tfrac{21}{16}.$$

I.e. the locus is a circle whose center is $(\tfrac{3}{4}, -\tfrac{3}{2})$ and whose radius is $\tfrac{1}{4}\sqrt{21}$.

28. The preceding examples show that the locus of a quadratic function involving two variables is a circle, if the function does not contain xy and if the coefficients of x^2 and y^2 are equal.

EXERCISE 8

Solve graphically:

1. $\begin{cases} x^2+y^2=4, \\ x+y=3. \end{cases}$

2. $\begin{cases} x^2+y^2=16, \\ x-y=4. \end{cases}$

3. $\begin{cases} x^2+y^2=50, \\ x-y=-6. \end{cases}$

4. $\begin{cases} x^2+y^2=9, \\ x-2y=2. \end{cases}$

5. $\begin{cases} x^2+y^2=16, \\ 2y-3x=6. \end{cases}$

6. $\begin{cases} x^2-2x+y^2-4y=0, \\ y=2x. \end{cases}$

7. $\begin{cases} x^2-4x+y^2+2y+3=0, \\ x-y=3. \end{cases}$

8. $\begin{cases} x^2-10x+y^2=0, \\ x^2+6x+y^2=16. \end{cases}$

9. $\begin{cases} (x+1)^2-(y-1)^2=2, \\ (x-1)^2+(y+1)^2=8. \end{cases}$

10. $\begin{cases} x^2+y^2=1, \\ (x-1)^2+y^2=2. \end{cases}$

PART II

SOLUTION OF EQUATIONS BY MEANS OF STANDARD CURVES

29. A disadvantage of the preceding graphic methods is the fact that they often require a great deal of numerical calculation, and that the necessary curves are difficult to draw. In the following chapters, methods will be given for the solution of quadratics, cubics, and biquadratics by means of one standard curve, and straight lines or circles; *i.e.* one curve may be used to solve all quadratics or all cubics, etc. The construction of these curves requires very little calculation, and once constructed, each curve may be used for the solution of many problems.

Three curves are used in the following chapters, viz. a **parabola** $y = x^2$, a **cubic parabola** $y = x^3$, and an **equilateral hyperbola** $y = \frac{1}{x}$.

$y = x^2$ was drawn and discussed in § 8.

A locus of the form $y = \frac{a}{x}$ was given in § 25, Ex. 2, and the graph of $y = x^3$ will be given in § 49.

Any one of these three curves may be used to solve with rules and compasses either quadratics or cubics, but only the parabola and equilateral hyperbola yield simple solutions for biquadratics.

CHAPTER V

QUADRATIC EQUATIONS

30. To solve the quadratic

$$ax^2 + bx + c = 0 \qquad (1)$$

by means of a standard curve, we split the equation (1) into two simultaneous equations, one of which is the standard curve, while the other is a straight line or circle.

Thus, if $\qquad ax^2 + bx + c = 0, \qquad (1)$

Let $\qquad \left.\begin{array}{r} y = x^2. \qquad (2) \\ \text{Substituting in (1),} \quad ay + bx + c = 0. \qquad (3) \end{array}\right\}$

The solution of the system (2), (3) for x produces the required roots of (1).

But the graph of (3) is a straight line, while the graph of (2) is identical for all quadratic equations. Hence, after the graph $y = x^2$ (see annexed diagram) has been constructed, any quadratic equation may be solved by the construction of a straight line, provided the roots lie within the limits of the represented abscissas (-6 and $+6$).

Ex. 1. Solve $11x^2 + 30x - 165 = 0. \qquad (1)$

Let $\qquad y = x^2. \qquad (2)$

Then $\qquad 11y + 30x - 165 = 0. \qquad (3)$

In (3), if $x = 0$, then $y = 15$; if $y = 0$, then $x = 5\frac{1}{2}$. The straight line joining the points (0, 15) and ($5\frac{1}{2}$, 0) is the graph of (3), which intersects the graph of (2) in P and P'. By measuring the abscissas of P and P', we have

$$x = 2.7, \text{ or } x = -5.5.$$

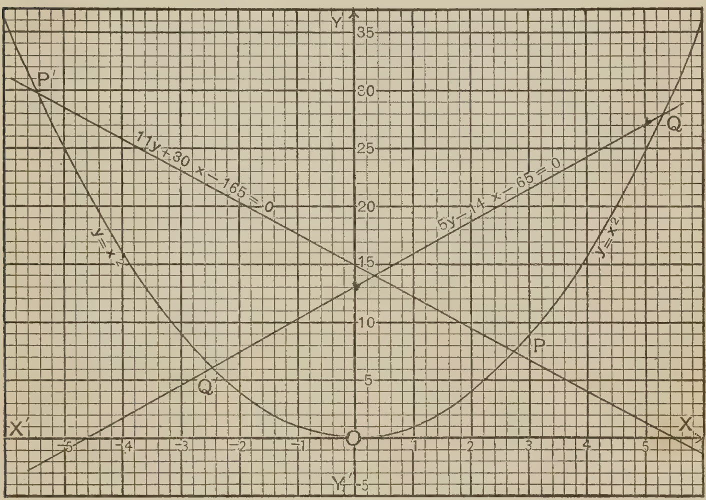
Ex. 2. Solve $5x^2 - 14x - 65 = 0. \qquad (1)$

Let $\qquad y = x^2. \qquad (2)$

Then $\qquad 5y - 14x - 65 = 0. \qquad (3)$

Locating two points of the equation (3), *e.g.* (0, 13) and (5, 27), and joining by a straight line produces the graph of (3), which intersects the graph of (2) in Q and Q'. Measuring the abscissas of Q and Q', we obtain

$$x = 5.3, \text{ or } x = -2.5.$$



31. In the equation $ay + bx + c = 0$, if $x = 0$, then $y = -\frac{c}{a}$, and if $y = 0$, then $x = -\frac{c}{b}$. Hence, by laying off on the x-axis the distance $-\frac{c}{b}$ and on the y-axis the distance $-\frac{c}{a}$, and applying a straight edge, the roots of the equation $ax^2 + bx + c = 0$ can frequently be determined by inspection.

If the two points constructed on the axes lie very closely together, the drawing is likely to be inaccurate, and it is better to locate one or both points outside the axes.

EXERCISE 9

Solve the following equations by the graphical method:

1. $x^2-x-6=0$.
2. $x^2+x-2=0$.
3. $x^2-3x-18=0$.
4. $x^2+3x-10=0$.
5. $x^2-2x-8=0$.
6. $x^2+2x-4=0$.
7. $x^2-5x-15=0$.
8. $4x^2-25x+20=0$.
9. $3x^2+20x+12=0$.
10. $x^2+x-5=0$.
11. $x^2-2x-9=0$.
12. $3x^2+7x-42=0$.
13. $2x^2+5x-20=0$.
14. $5x^2-4x-5=0$.

32. Solution for large roots. By changing the unit of the abscissas and the unit of the ordinates, the same diagram may be used to represent $y=x^2$ for various values of x. For in the diagram we may assign any values to the abscissas, provided the corresponding ordinates are made equal to the squares of the abscissas. Thus after the graph of $y=x^2$ has been drawn from $x=-10$ to $x=10$, we may multiply the numbers on the x-axis by any number, *e.g.* 3, and thereby extend the diagram from $x=-30$ to $x=30$, provided we multiply the numbers on the y-axis by 3^2, or 9.

This change of scale units does not affect the character of the locus $ay+bx+c=0$, for this equation is a straight line whether the abscissas and ordinates are drawn on the same or different scales.

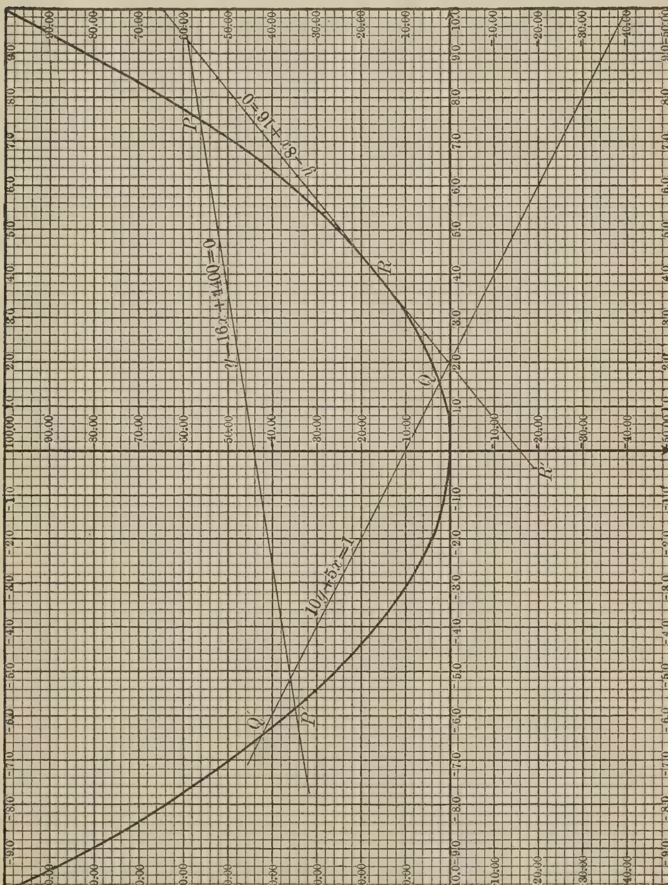
The annexed diagram can be used directly for roots between -10 and $+10$. If the roots are larger, but lie between -100 and $+100$, multiply the units by 10 and 10^2 respectively; *i.e. omit the decimal points* in the diagram.

For roots still larger, add another cipher to the values of the abscissas and two ciphers to the values of the ordinates.

Ex. 1. Solve graphically $x^2-16x-4400=0$.

Let $$y=x^2;$$
then $$y-16x-4400=0.$$

Obviously the regular diagram does not contain the required roots. Hence multiply the values of the abscissas by 10 and the values of the ordinates by 10^2; *i.e. disregard the decimal points.*

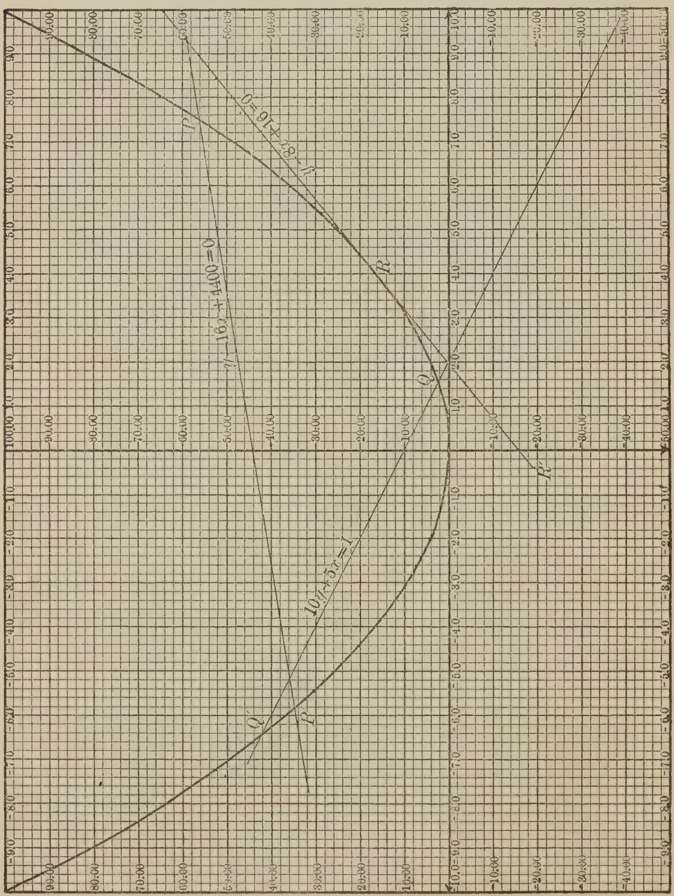


Since $\frac{c}{b}$ is very large, locate two points as follows:

If $\qquad x=0, \quad y=4400.$

If $\qquad x=100, \; y=6000.$

The line joining (0, 4400) and (100, 6000) intersects $y=x^2$ in



P and P'. By measuring the abscissas of P and P', we obtain $x = 74^+$ and $x = -59^+$.

33. Small roots. For small roots multiply the values of the abscissas by a fraction, most conveniently by .1, and the values of the ordinates by .01; *i.e.* place the decimal point in front of each number given in the diagram (except $x = 10$ and $y = 100$, which become 1.0 and 1.00 respectively).

Thus, 1.0, 2.0, 3.0, etc., become .10, .20, .30, etc. As this shifting of the decimal point is a simple operation, it may be done mentally, without any actual alterations of the numbers in the diagrams.

Ex. 2. Solve $10x^2 + 5x = 1$.

Let	$y = x^2$.	
Then	$10y + 5x = 1$.	
If	$x = 0,$	$y = .1$.
If	$x = 1,$	$y = -.4$.

Since in the original diagram such small fractions of y cannot be well represented, multiply the numbers on the x-axis by .1 and the numbers on the y-axis by .01; *i.e.* imagine the decimal point to be placed in front of each number.

Then the straight line that joins (0, .1) and (1, $-$.4) intersects the parabola in Q and Q'. The measurement of the abscissas of Q and Q' gives the roots

$$x = .15, \text{ or } x = -.65.$$

NOTE. The student should draw a diagram similar to the one used in the text, but on a larger scale. The cross-section paper employed should have each unit divided into 10 parts.

EXERCISE 10

Solve graphically:

1. $x^2 - 15x - 4500 = 0$.
2. $x^2 - 10x - 3000 = 0$.
3. $x^2 + 80x + 1200 = 0$.
4. $x^2 + 40x = 1200$.
5. $x^2 + 30x = 4000$.
6. $x^2 + 80x = -700$.
7. $x^2 - 10x - 600 = 0$.
8. $x^2 + 8x - 128 = 0$.
9. $x^2 - 30x - 1800 = 0$.
10. $x^2 + 33x - 1210 = 0$.

11. $2x^2 + 3x - 1500 = 0.$
12. $3x^2 + 10x = 3000.$
13. $x^2 + 29x = 210.$
14. $3x^2 + 200x - 1200 = 0.$
15. $50x^2 - 15x - 6 = 0.$
16. $10x^2 - 6x - 1 = 0.$
17. $4x^2 + 5x - 1 = 0.$
18. $20x^2 + 3x - 1 = 0.$
19. $50x^2 - 5x - 3 = 0.$
20. $25x^2 + 10x - 3 = 0.$
21. $50x^2 + 5x - 1 = 0.$
22. $8x^2 - 2x - 1 = 0.$

34. Graphic representation of a quadratic function.

Consider the equation

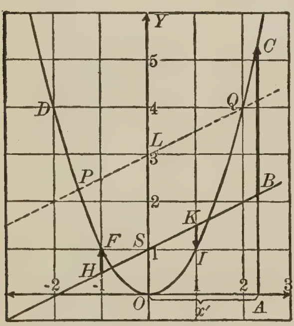
$$x^2 + px + q = 0. \quad (1)$$

Let $$y = x^2. \quad (2)$$

Then $$y + px + q = 0, \quad (3)$$

or $$y = -px - q.$$

In the annexed diagram, let COD represent the parabola $y = x^2$, and BH the straight line $y + px + q = 0$, or $y = -px - q$.



Let OA or x' be any particular value of x,

then $$CA = x'^2,$$
and $$BA = -px' - q.$$

Hence

$$CB = CA - BA = x'^2 + px' + q.$$

I.e. *the value of the function* $x^2 + px + q$ *for any particular value* x' *is represented by that part of the corresponding ordinate which is intercepted between the straight line* $y + px + q = 0$, *and the parabola* $y = x^2$. The distance is measured from the straight line, and is taken positive if it extends upward, negative if it extends downward.

Thus, in the annexed diagram,

$y = x^2 - \frac{1}{2}x - 1$, and we have:

If $x = -1$, $y = HF = \frac{1}{2}$.
If $x = 1$, $y = KI = -\frac{1}{2}$.
If $x = 2$, $y = 2$, etc.

NOTE. If we consider the distances cut off from SB by the ordinates, as abscissas, *e.g.* $SK = x = 1$, then the parabola represents the function $x^2 + px + q$ in so-called "oblique coördinates."

Ex. 1. Find the values of x which will make the function $x^2 - \frac{1}{2}x - 1$ equal to 2, *i.e.*

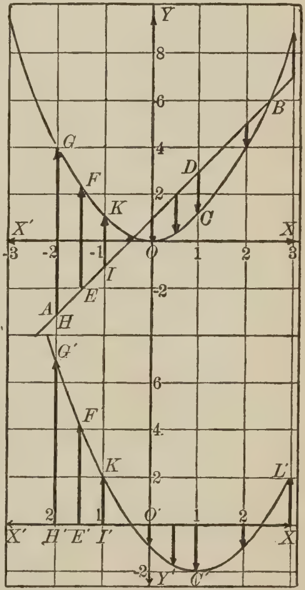
$$x^2 - \tfrac{1}{2}x - 1 = 2.$$

On YY' lay off $SL = 2$, and through L draw $PQ \parallel BE$, meeting the parabola in P and Q. By measuring the abscissas of P and Q we find

$$x = -\tfrac{3}{2}, \text{ or } 2.$$

Ex. 2. Find the smallest value of the function $x^2 - 2x - 1$.

Construct AB, the locus of $y - 2x - 1 = 0$. (See next diagram.) Draw a tangent parallel to AB, touching the parabola in C; then $DC = -2$ is the required value.



35. To construct the graph of $x^2 + px + q$ in the usual manner (rectangular coördinates), make $H'G' = HG, E'F' = EF, I'K' = IK$, etc. The curve $G'F'C'L'$ is the required locus of $x^2 + px + q$ (*i.e.* of $x^2 - 2x - 1$).

NOTE. § 35 makes it possible to construct the locus of a quadratic function without any computation.

36. *The value of the function $ax^2 + bx + c$ is equal to a times the part of the corresponding ordinate which is intercepted by the straight line $ay + bx + c = 0$, and the parabola $y = x^2$.*

For $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

But the function $x^2 + \frac{b}{a}x + \frac{c}{a}$ is represented by that part of the ordinate that lies between $y = x^2$ and $y + \frac{b}{a}x + \frac{c}{a} = 0$ or $ay + bx + c = 0$. Hence $ax^2 + bx + c$ is equal to a times this intercept.

EXERCISE 11

Find graphically:

1. The value of $x^2 - 2x - 2$, if x equals -3, -2, .5, $1\frac{1}{2}$.
2. The value of $x^2 + x - 3$, if x equals -1.5, -1, 0, 2.
3. The value of $x^2 - 5x - 12$, if x equals -2.1, -1.5, 3.5.
4. The value of $x^2 + 4x + 5$, if x equals -7.5, 9.4, -8.8.
5. The values of x if $x^2 - 2x - 10 = 4$.
6. The values of x if $x^2 + 10x + 10 = 5$.
7. The smallest value of $x^2 - 4x + 3$.
8. The smallest value of $x^2 + 10x - 5$.
9. The smallest value of $x^2 + 7x - 3$.
10. The smallest value of $x^2 - 5x + 2$.
11. The value of $2x^2 + 6x + 7$, if x equals -3, 6.5, 7.

Without calculating the various values of the function, construct the loci of:

12. $x^2 + 6x + 10$.
13. $x^2 + 4x - 5$.
14. $x^2 - 4x + 7$.
15. $x^2 + 5x - 3$.
16. $x^2 - 3x + 7$.
17. $x^2 + x + 1$.
18. $x^2 + 4x - 7$.

37. Equal roots. If the line $ay + bx + c = 0$ is a tangent to the parabola $y = x^2$, the two points of intersection coincide, and the two roots of $ax^2 + bx + c = 0$ are equal.

Ex. 1. Solve $x^2 - 8x + 16 = 0$.

Let $\qquad y = x^2.$

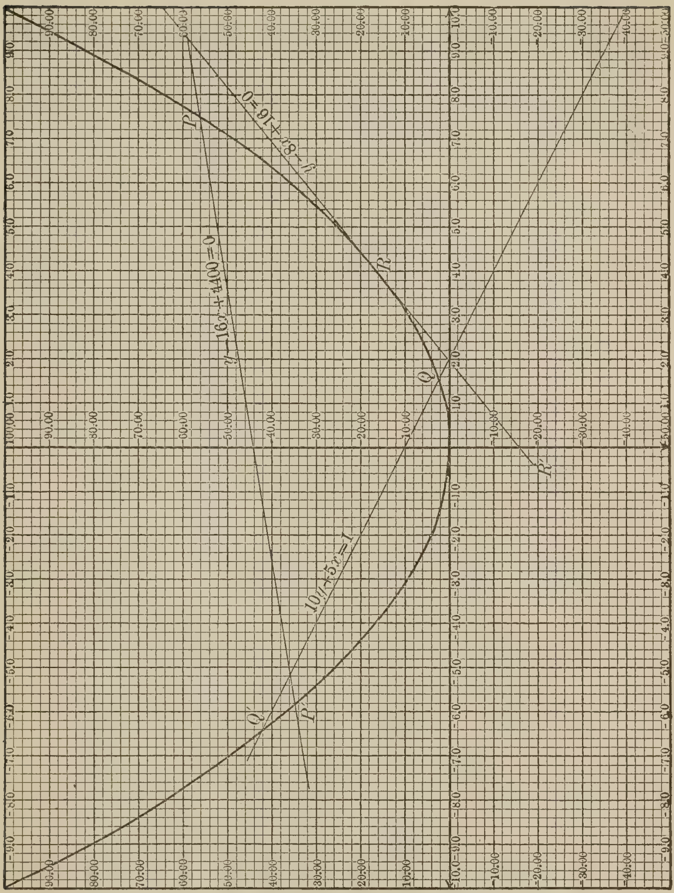
Then $\qquad y - 8x + 16 = 0.$

If $\qquad x = 0,\ y = -16.$

If $\qquad x = 10,\ y = 64.$

The line RR', which joins $(0, -16)$ and $(10, 64)$, is tangent to the parabola at the point R.

Measuring the abscissa of R, we obtain two equal roots, 4 and 4.



38. If the roots are equal or nearly equal, the graphic method is, however, liable to be inaccurate, since a slight inaccuracy in the construction of $ay + bx + c = 0$ usually produces a considerable error in the value of x.

39. Complex roots. If the line $ay+bx+c=0$ does not intersect the parabola $y=x^2$, the roots are complex, and their values may be found by means of the following theorems.

40. Let $x^2+px+q=0$ represent an equation *whose roots are equal;* then these roots are, by the general formula:

$$-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}.*$$

Hence
$$\frac{p^2}{4}-q=0.$$

Assuming that d is a positive quantity, it can easily be shown that

$$x^2+px+q-d=0 \text{ has the roots } -\frac{p}{2}\pm\sqrt{d}; \qquad (1)$$

$$x^2+px+q \quad =0 \text{ has the equal roots } -\frac{p}{2}, -\frac{p}{2}; \qquad (2)$$

$$x^2+px+q+d=0 \text{ has the roots } -\frac{p}{2}\pm\sqrt{-d}. \qquad (3)$$

The roots of (3) are complex and cannot be found directly by the graphic method, but if we solve (1) instead, we only have to multiply the irrational parts of the answer by $\sqrt{-1}$ to obtain the roots of (3). The straight line which serves to solve (1) can be obtained from the one which solves (3) by means of the following proposition.

41. If $x^2+px+q=0$ *has equal roots, the three straight lines*

$$y+px+q-d=0, \qquad (1\,a)$$

$$y+px+q \quad =0, \qquad (2\,a)$$

$$y+px+q+d=0, \qquad (3\,a)$$

are parallel, and the second one is equidistant from the other two.

These lines (AB, CD, and EF in annexed diagram) are parallel by § 21.

By making $x=0$, we obtain:

$$OA=-q+d,$$
$$OC=-q,$$

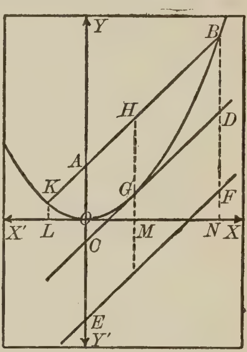
* Schultze's Algebra, p. 269.

$$OE = -q - d.$$

Hence $$AC = CE = d;$$

I.e. CD is equidistant from AB and EF.

42. Hence if EF is known, draw the tangent $CD \parallel EF$, make $CA = EC$, and construct $AB \parallel EF$; then AB is the required line which produces the roots of (1).



43. The construction, however, is simplified by the following theorems:

1. *The abscissa of the point of contact (G) is equal to the rational part of the roots of* (1) *and* (3).

For this abscissa $= -\frac{p}{2}$.

2. *A parallel to YY' through the point of contact (G) bisects the chord KB, and hence any chord parallel to the tangent.*

For the abscissas of K, H, and B are respectively

$$OL = -\frac{p}{2} - \sqrt{d};$$

$$OM = -\frac{p}{2};$$

$$ON = -\frac{p}{2} + \sqrt{d}.$$

Hence $$LM = MN = \sqrt{d}.$$

Therefore, according to a geometric theorem,* $KH = HB$.

44. Graphic solution for complex roots. *To solve the equation*

$$x^2 + bx + c = 0, \qquad (1)$$

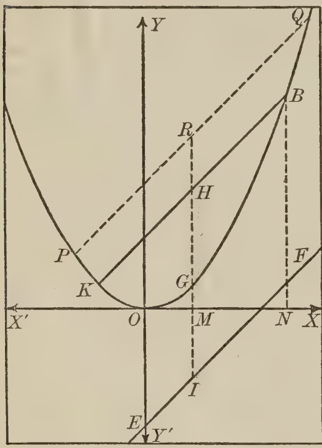
which has imaginary roots.

Construct the locus EF of $y + bx + c = 0$, and draw any chord $PQ \parallel EF$. (See diagram, page 38.)

Through R, the midpoint of PQ, draw $RI \parallel YY'$ intersecting

* Schultze and Sevenoak's Geometry, § 144.

the parabola in G, and EF in I. Make $GH = IG$, and through H draw $KB \parallel EF$.



Then the abscissa of H is the real part, and the difference of the abscissas of B and H multiplied by $\sqrt{-1}$ is the imaginary part of the required roots; *i.e.* $x = \overline{OM} \pm \overline{MN} \times \sqrt{-1}$.

Note. The line RI may also be constructed by drawing an ordinate through one point $\left(0, -\frac{b}{2a}\right)$, or if $a = 1$, through $\left(0, -\frac{b}{2}\right)$.

45. If the coefficient of x^2 is a, the solution is the same as if this coefficient was unity; for by dividing $ax^2 + bx + c = 0$ by a, we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \qquad (2)$$

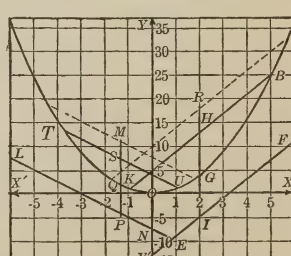
The straight line which serves to solve (2) is therefore

$$y + \frac{b}{a}x + \frac{c}{a} = 0,$$

or $$ay + bx + c = 0;$$

i.e. we may substitute $y = x^2$ directly into the given equation.

Ex. 1. Solve $x^2 - 4x + 13 = 0$.



Let $y = x^2$.
Then $y - 4x + 13 = 0$.
If $x = 0, y = -13$.
If $x = 5, y = 7$.

Join $(0, -13)$ and $(5, 7)$ by line EF, and through the midpoint (R) of any parallel chord draw $RI \parallel YY'$. Make $GH = IG$ and draw $KB \parallel EF$.

By measuring we obtain:
The abscissa of $H = 2$.

The difference of the abscissas of B and $H = 3$.

Hence the required roots are

$$x = 2 \pm 3\sqrt{-1} \text{ or } 2 \pm 3i.$$

Ex. 2. Solve $2x^2 + 5x + 15 = 0$. (1)

Let $y = x^2$. (2)

Then $2y + 5x + 15 = 0$. (3)

Construct the line (3), *i.e.* LN, and through the midpoint (M) of any parallel chord draw $MP \parallel YY'$. Make $QS = PQ$ and draw $SU \parallel LN$. The roots produced by TU are -1.3 ± 2.4. Hence the required roots are $-1.3 \pm 2.4\sqrt{-1}$, or $-1.3 \pm 2.4i$.

EXERCISE 12

Solve the following equations graphically:

1. $x^2 - 10x + 25 = 0$.
2. $x^2 - 6x + 13 = 0$.
3. $x^2 + 4x + 8 = 0$.
4. $x^2 + 8x + 20 = 0$.
5. $x^2 - 8x + 25 = 0$.
6. $x^2 - 10x + 29 = 0$.
7. $x^2 + 7x + 21 = 0$.
8. $x^2 - 5x + 15 = 0$.
9. $x^2 + 3x + 27 = 0$.
10. $x^2 + 9x + 36 = 0$.
11. $x^2 + x + 1 = 0$.
12. $x^2 + 2x + 1 = 0$.
13. $2^2x + 2x + 3 = 0$.
14. $4x^2 - 12x + 25 = 0$.

46.* Solution of quadratic equations by means of the standard curve $y = \frac{1}{x}$.

As stated in § 29, the parabola $y = x^2$ is not the only curve that may be used for the graphic solution of quadratic equations by means of straight lines. A curve that gives a very convenient solution is the equilateral hyperbola $y = \frac{1}{x}$, which is plotted in the annexed diagram. It consists of two disconnected branches which approach the axes indefinitely.

NOTE. To plot this curve exactly, it is necessary to locate several points between $x = 0$ and $x = 1$. Thus, if $y = 2$, $x = \frac{1}{2}$; if $y = 3$, $x = \frac{1}{3}$; if $y = 4$, $x = \frac{1}{4}$; etc. (See table on page 84.)

* Paragraphs marked by asterisk * may be omitted.

47.* To solve the equation

$$ax^2 + bx + c = 0. \qquad (1)$$

Let $$y = \frac{1}{x}, \text{ or } x = \frac{1}{y}. \qquad (2)$$

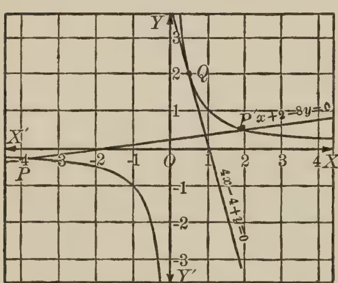
Partly substituting this value for x in equation (1),

$$\frac{ax}{y} + \frac{b}{y} + c = 0.$$

Or $$\left.\begin{aligned} ax + b + cy &= 0. \qquad (3) \\ y &= \frac{1}{x}. \qquad (2) \end{aligned}\right\}$$

The solution of the system (2), (3) for x produces the required roots of (1).*

Ex. 1. Solve $x^2 + 2x - 8 = 0$.



Let $$y = \frac{1}{x}.$$

Then $$x + 2 - 8y = 0.$$

If $y = 0$, $x = -2$; if $y = 1$, $x = 6$.

The straight line that joins $(-2, 0)$ and $(6, 1)$ intersects (2) in P and P'. Measuring the abscissas of P and P', we obtain $x = -4$ or $+2$.

If the line $ax + b + cy = 0$ is tangent to $y = \frac{1}{x}$, the roots are equal.

Ex. 2. Solve $4x^2 - 4x + 1 = 0$.

If $$y = \frac{1}{x}, \qquad (2)$$

then $$4x - 4 + y = 0. \qquad (3)$$

The line (3) touches (2) at Q. Hence there are two equal roots, $\frac{1}{2}$ and $\frac{1}{2}$.

* This method may be used for all equations of the form $ax\,f(x) + bf(x) + c = 0$.

Let $$y = \frac{1}{f(x)}.$$

Then $$ax + b + cy = 0.$$

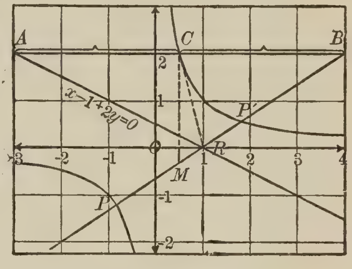
48.* **Complex roots** can be found by a method similar to the one given in § 44.

Students who wish to derive this method may be guided by the following suggestions:

1. Consider the same equations as in § 40.

2. These equations are represented by the lines

$$x + p + (q - d)\,y = 0, \qquad (1\,a)$$
$$x + p + qy = 0, \qquad (2\,a)$$
$$x + p + (q + d)\,y = 0. \qquad (3\,a)$$



3. Instead of being parallel (as in § 41) the lines (1a), (2 a), and (3 a) meet in a point (R) on the x-axis whose abscissa is $-p$.

4. The lines (1 a), (2 a), (3 a) intercept equal parts on any line parallel to the x-axis.

5. A parallel to the y-axis through the midpoint of OR intersects $y = \dfrac{1}{x}$ at C, the point of contact of (2 a).

The annexed diagram solves the equation $x^2 - x + 2 = 0$. The line $x - 1 + 2\,y = 0$, or RA, does not intersect the curve, but the corresponding line RB produces the roots $.5 \pm 1.3$. Hence the required roots are $.5 \pm 1.3 \times \sqrt{-1}$.

EXERCISE 13

Solve by means of the equilateral hyperbola the following equations:

1. $x^2 - 2\,x - 15 = 0$.
2. $x^2 - x - 6 = 0$.
3. $x^2 - 6\,x + 5 = 0$.
4. $x^2 + 5\,x + 4 = 0$.
5. $x^2 - 2\,x + 10 = 0$.
6. $x^2 + 6\,x + 6 = 0$.

[For more examples see Exs. 9 and 12.]

NOTE. The solution of quadratics by means of the cubic parabola $y = x^3$ is given in § 60.

CHAPTER VI

CUBIC EQUATIONS

49. Solution of incomplete cubics. To solve an incomplete cubic of the form $ax^3 + bx + c = 0$, the method that was used for quadratics (§ 30) may be employed.* Thus, to solve

$$ax^3 + bx + c = 0, \tag{1}$$

let $$\left.\begin{aligned} y &= x^3. \\ ay + bx + c &= 0. \end{aligned}\right\} \quad \begin{matrix}(2)\\(3)\end{matrix}$$
Then

The solution of the system (2), (3) for x produces the required roots.

But the graph of (3) is a straight line, while the graph of (2) is a cubic parabola which is identical for all cubic equations. Hence after the graph of the cubic parabola (AOP in the diagram) has been constructed, any cubic may be solved by the construction of a straight line.

Ex. 1. Solve $4x^3 - 39x + 35 = 0.$ (1)

Let $y = x^3.$ (2)

Then $4y - 39x + 35 = 0.$ (3)

In (3), if $x = 0$, then $y = -8\frac{3}{4}$, and if $x = 4$, then $y = 30\frac{1}{4}$. The line joining $(0, -8\frac{3}{4})$ and $(4, 30\frac{1}{4})$ intersects the graph of (2) in P, P', and P''. By measuring the abscissas of P, P', and P'', we find $x = -3\frac{1}{2}$, or $+1$, or $2\frac{1}{2}$.

50. In the equation $ay + bx + c$, if $x = 0$, then $y = -\frac{c}{a}$; if $y = 0$, then $x = -\frac{c}{b}$. Hence, by taking on the x-axis the point $-\frac{c}{b}$, on the y-axis the point $-\frac{c}{a}$, and applying a straight edge, the roots of the equation $ax^3 + bx + c = 0$ can frequently be

* This method may be used for any equation of the form $af(x) + bx + c = 0$; *e.g.* $ax^5 - bx + c = 0$, or $x - e \sin x = 0$, etc.

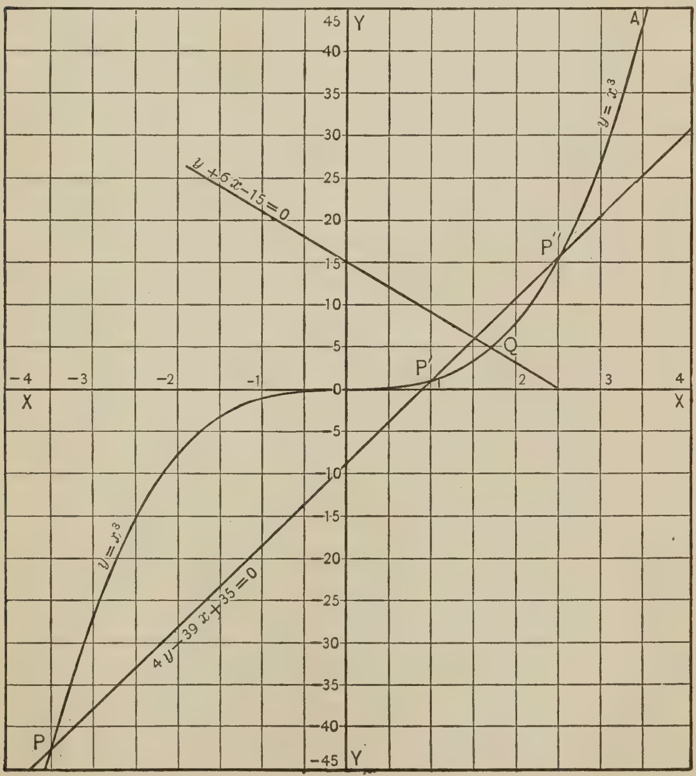
determined by inspection. If the two points thus constructed on the axes lie very closely together, the drawing is liable to be inaccurate, and it is better to locate one or both points outside the axes.

Ex. 2. Solve $x^3+6x-15=0$. (1)

Let $$y=x^3. \quad (2)$$

Then $$y+6x-15=0. \quad (3)$$

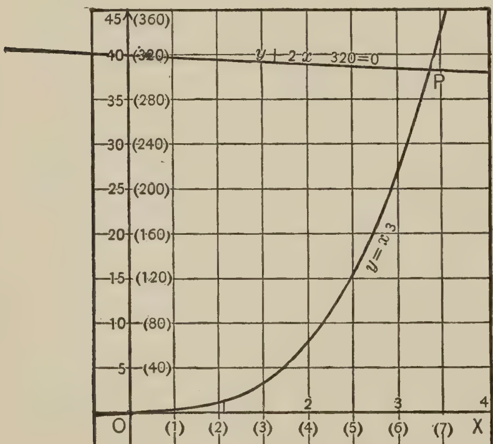
Hence, the distances cut off by (3) on the x- and y-axes are respectively $2\frac{1}{2}$ and 15, and the line (3) is easily constructed. As there is only one point of intersection, Q, the equation has only one real root, viz. 1.7^+.



51. Solution for large roots. One diagram may be used for the solution of large and small roots. For in the diagram we may assign any values to the abscissas, provided the corresponding ordinates are the cubes of the abscissas.

Thus, after the cubic parabola $y = x^3$ has been drawn, we may multiply the numbers on the x-axis by any convenient number, *e.g.* 3, provided we multiply the values of the ordinates by the cube of the number, *i.e.* 27.

Similarly, to find small roots, multiply the values of the abscissas by a fraction *e.g.* $\frac{1}{2}$, and the values of the corresponding ordinates by the cube of this fraction, *i.e.* $\frac{1}{8}$.



Ex. 3. Solve graphically $x^3 + 2x - 320 = 0$. (1)

Let $y = x^3$. (2)

Then $y + 2x - 320 = 0$. (3)

If $x = 0$, $y = 320$, and if $x = 8$, $y = 304$.

Obviously the preceding diagram cannot contain the roots, and the position of (3) shows that there cannot be a negative root.

Hence, multiply the values of the abscissas in the diagram by 2. Then the values of the ordinates must be multiplied by 8. (The resulting values are given in parenthesis.)

Joining the points (0, 320) and (8, 304), we obtain the real root 6.8^-, while the other roots are imaginary.

NOTE. The student should draw the graph of $y = x^3$ from $x = -3\frac{1}{2}$ to $x = +3\frac{1}{2}$ (or from -4 to $+4$) on a large scale, and use one curve for the solution of a number of equations. The table on page 84 will be found useful for the construction.

EXERCISE 14

Find graphically the real roots of the following equations:

1. $x^3+4x-16=0$.
2. $x^3-5x-12=0$.
3. $x^3-2x+4=0$.
4. $2x^3-9x+27=0$.
5. $x^3-7x+6=0$.
6. $4x^3-39x-35=0$.
7. $x^3-5x+20=0$.
8. $x^3-5x-15=0$.
9. $x^3-5x-5=0$.
10. $x^3-32x-80=0$.
11. $2x^3-5x+20=0$.
12. $x^3+8x-64=0$.
13. $x^3-10x-48=0$.
14. $x^3-9x+54=0$.
15. $x^3-14x+24=0$.
16. $x^3-30x-18=0$.
17. $x^3+10x-13=0$.
18. $x^3-45x-152=0$.
19. $x^3-60x+180=0$.
20. $x^3-90x+340=0$.
21. $x^3-75x-250=0$.
22. $x^3-100x+500=0$.
23. $x^3+120x-560=0$.
24. $x^3-200x+1200=0$.

52. Graphic representation of a cubic function.

Consider the equation

$$x^3+px+q=0. \tag{1}$$

Let

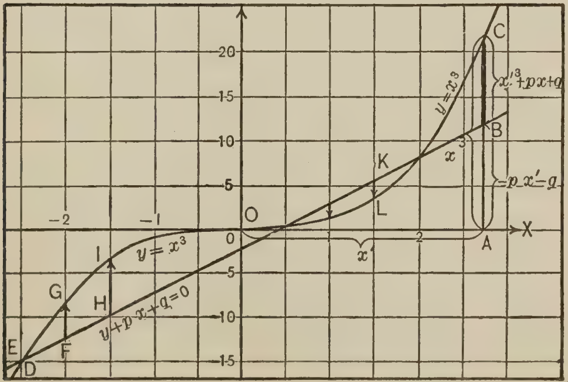
$$y=x^3. \tag{2}$$

Then

$$y+px+q=0, \tag{3}$$

or

$$y=-px-q.$$



In the annexed diagram, let COD represent the cubic parabola $y=x^3$, and BE the straight line $y+px+q=0$, or $y=-px-q$.

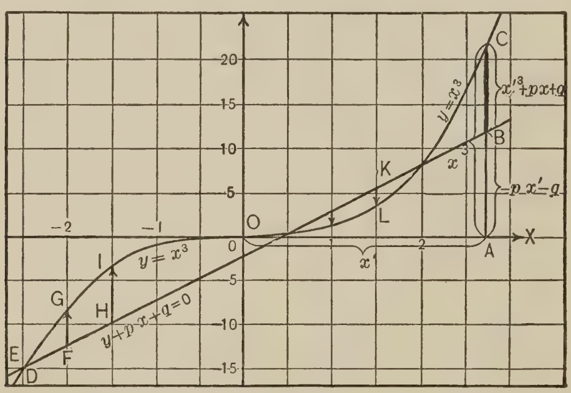
Let OA or x' be any particular value of x.

Then $\qquad CA = x'^3$,

and $\qquad BA = -px' - q$.

Hence $\qquad CB = CA - BA = x'^3 + px' + q$.

I.e. *the value of the function* x^3+px+q *for any particular value* x' *is represented by that part of the corresponding ordinate which is intercepted between the straight line* $y+px+q=0$, *and*



the cubic parabola $y=x^3$. The distance is measured from the straight line, and is taken positive if it extends upward, negative if it extends downward.

Thus in the annexed diagram $y=x^3-\frac{21}{4}x+\frac{5}{2}$, and we have

$$\text{if } x=-2,\ y=FG=5,$$
$$x=-1\tfrac{1}{2},\ y=HI=7,$$
$$x=1\tfrac{1}{2},\ y=KL=-2, \text{ etc.}$$

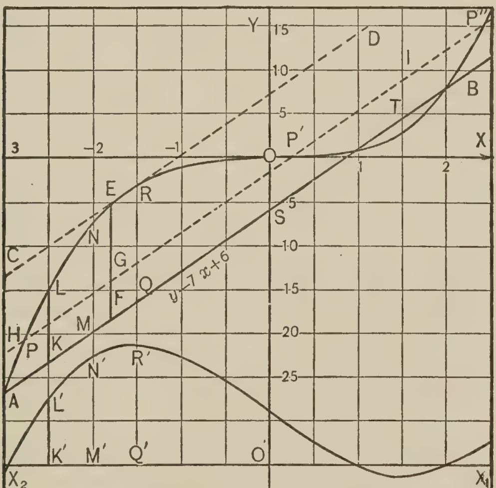
Ex. 1. Find the greatest value of the function x^3-7x+6, for a negative x.

Construct AB, the locus of $y - 7x + 6 = 0$. Draw CD parallel to AB, touching the cubic parabola in E; then FE, or 14, is the required value.

Ex. 2. Which values of x will make the function $x^3 - 7x + 6$ equal to 4, *i.e.*

$$x^3 - 7x + 6 = 4?$$

On any ordinate, from the straight line AB, lay off 4 units upward, as FG. Through G draw HI parallel to AB, intersecting the cubic parabola in P, P', and P''. By measuring the abscissas of P, P', and P'', we find $x = -2\frac{3}{4}$, or $\frac{1}{4}$, or $2\frac{1}{2}$.



NOTE. If we consider the distances cut off from SB by the ordinates, as abscissas, *e.g.* $ST = 1\frac{1}{2}$, then the cubic parabola represents the function $x^3 + px + q$ in so-called "oblique coördinates."

53. To construct the graph of $x^3 + px + q$ in the usual manner (rectangular coördinates), make $K'L' = KL$, $M'N' = MN$, $O'R' = OR$, etc. The curve $L'N'R'$ is the required graph of $x^3 + px + q$.

54. *The value of the function $ax^3 + bx + c$ is equal to a times the part of the corresponding ordinate which is intercepted by the straight line $ay + bx + c = 0$, and the cubic parabola $y = x^3$.*

The proof is similar to that of § 36.

EXERCISE 15

Find graphically:

1. The value of $x^3 + 4x - 16$, if x equals -3, -2.5, -2.1, 3.5.
2. The value of $x^3 + 4x - 8$, if x equals -1.6, -1.5, 2, 1.5.
3. The value of $x^3 - 6x - 15$, if $x = -3$, -2, 1.5, 3.5.
4. The value of $x^3 - 5x + 18$, if $x = -8$, -5, $+3$, $+7$.
5. The value of x, if $x^3 - 5x - 12 = 5$.
6. The value of x, if $x^3 - 5x - 12 = -10$.
7. The value of x, if $x^3 - 5x - 12 = -40$.
8. The value of x, if $x^3 - 5x - 12 = 10$.
9. The smallest value of $x^3 - 5x - 12$ for a positive x.
10. The greatest value of $x^3 - 5x + 10$ for a negative x.
11. Construct the graph of $x^3 - 12x - 30 = 0$.
12. Construct the graph of $x^3 - 8 = 0$.

Find:

13. The value of $2x^3 + 9x + 20 = 0$, if x equals 3, 2.5, -1.5.
14. The value of $3x^3 + 9x - 25 = 0$, if x equals -3, -5, -2.
15. The smallest value of $3x^3 - 9x - 25$, for a positive x.

55. The preceding paragraphs may be used to locate the line $ay + bx + c = 0$ by determining two values of the function $ax^3 + bx + c$. In applying this method it is advisable to reduce the coefficient of x^3 to unity by dividing by a.

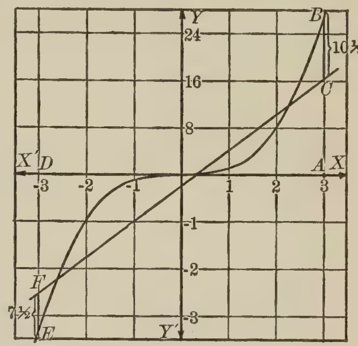
E.g., let $$2x^3 - 12x + 3 = 0.$$

Dividing by 2, $$x^3 - 6x + \tfrac{3}{2} = 0.$$

If $$x = 3,\ x^3 - 6x + \tfrac{3}{2} = 10\tfrac{1}{2}.$$

If $$x = -3,\ x^3 - 6x + \tfrac{3}{2} = -7\tfrac{1}{2}.$$

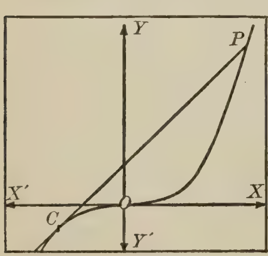
Through the point A (whose abscissa $= 3$) draw an ordinate meeting the cubic parabola in B, and on BA lay off downward $BC = 10\frac{1}{2}$. Similarly, through the point E (whose abscissa $= -3$) draw a perpendicular EF upward equal to $7\frac{1}{2}$; join FC, which is the required line.



56. Equal roots. If the line $ay + bx + c = 0$ is tangent to the cubic parabola, two points of intersection coincide, and two roots of the equation $ax^3 + bx + c = 0$ are equal.

The straight line must intersect the parabola at least once; hence every cubic equation has at least one real root.

57. It can be proved that the sum of the roots of an incomplete equation of the form $ax^3 + bx + c = 0$ is equal to zero. Hence if one root is m, and the other two are equal, then these equal roots are each $-\frac{m}{2}$; *i.e.* if the abscissa of $P = m$, the abscissa of C, the point of contact, equals $-\frac{m}{2}$.



Since it is difficult to locate graphically a point of contact with accuracy, it is advisable to determine equal roots by the preceding relation.

58. Complex roots of incomplete cubics. If the line $ay + bx + c = 0$ meets the cubic parabola in only one point, then two roots are complex. To find complex roots of the form $n \pm \sqrt{-t}$, we employ the same method as for

quadratic equations; viz. we determine the line that produces the roots $n \pm \sqrt{t}$.

If the equation $ax^3 + bx + c = 0$ has one root equal to m, the left member is divisible by $x - m$, and the equation may be represented in the form

$$a(x-m)(x^2+px+q)=0.$$

Supposing that $x^2+px+q=0$ *has equal roots*, and that d is a positive quantity, we consider the equations:

$$a(x-m)(x^2+px+q-d)=0, \tag{1}$$

$$a(x-m)(x^2+px+q) \qquad =0, \tag{2}$$

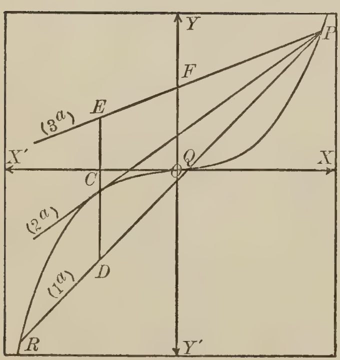
$$a(x-m)(x^2+px+q+d)=0. \tag{3}$$

In the same manner as in § 40 it follows that the roots of the equations (1), (2), and (3) are respectively

$$m, -\frac{p}{2} \pm \sqrt{d};$$

$$m, -\frac{p}{2}, -\frac{p}{2};$$

$$m, -\frac{p}{2} \pm \sqrt{-d}.$$



Hence the roots of (3) can be found by solving (1).

But the three straight lines (1^a), (2^a), and (3^a) which serve to solve (1), (2), and (3) respectively are connected by the following geometric relations:

1. *The three lines* (1^a), (2^a), *and* (3^a) *meet in a point* (m, m^3), *i.e.* P.

For m is a root of the three equations (1), (2), and (3).

2. *The three lines intercept equal parts on an ordinate drawn through the point of contact* C, *or* $DC = CE$.

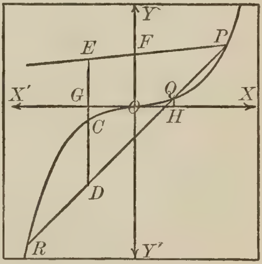
For according to § 52, CD is equal to the value of (3) if $x = -\frac{p}{2} = -\frac{m}{2}$ (§ 57), *i.e.* $CD = -\frac{md}{2}$.

Similarly, it follows from (1) that $CE = \frac{md}{2}$; *i.e.* CD and CE are equal and lie on opposite sides of C.

3. *The line* (2^a) *is tangent to the cubic parabola at* C.

This follows from § 56.

4. *The abscissas of* D, C, *and* E *are equal to* $-\frac{m}{2}$, *hence* $EF = \frac{1}{2}FP$ (§ 57).

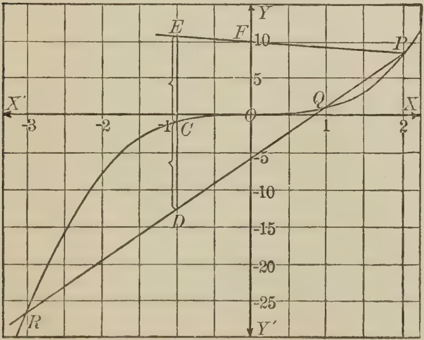


59. Construction of complex roots. Let $ax^3 + bx + c = 0$ have two complex roots.

Substitute $y = x^3$.

Then $$ay + bx + c = 0. \qquad (3)$$

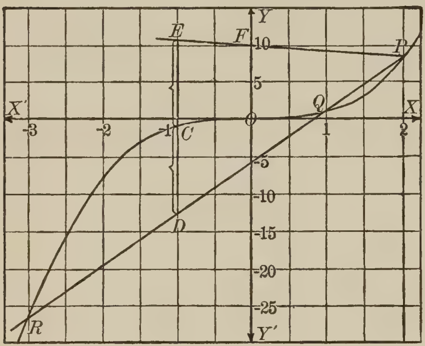
Construct PF, the locus of (3), and let it meet the parabola in one point, P, and the y-axis in F. Produce PF by one half its length to E, and through E draw an ordinate, meeting the cubic parabola in C. Produce EC by its own length to D and draw PD, intersecting the curve in Q and R. Then the abscissa of D is the real part, and the difference of the abscissas of Q and D is the imaginary part of the required roots; *i.e.* $x = \overline{OG} \pm \overline{GH}\sqrt{-1}$.



Ex. 1. Solve
$$x^3 + x - 10 = 0. \qquad (1)$$
Let $$y = x^3. \qquad (2)$$
Then $$y + x - 10 = 0. \qquad (3)$$

Construct the locus of (3), *i.e.* PF, which intersects the cubic parabola in one point, viz. P.

Hence the equation has one real root, which equals 2, and two imaginary roots.



Produce PF by one half its length to E. Through E draw an ordinate which meets the curve in C. Produce EC by its own length to D, and draw PD, intersecting the cubic parabola in Q and R.

The abscissa of $D = -1$, the difference of the abscissas of Q and $D = 2$.

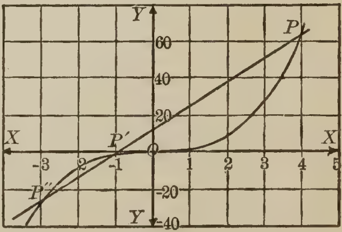
Hence the complex roots are $-1 \pm 2\sqrt{-1}$.

EXERCISE 16

Find the real and complex roots of the following equations:

1. $x^3 - 3x - 2 = 0$.
2. $x^3 - 3x + 2 = 0$.
3. $x^3 + x + 10 = 0$.
4. $4x^3 - 11x - 10 = 0$.
5. $4x^3 - 3x - 26 = 0$.
6. $x^3 + 9x + 26 = 0$.
7. $x^3 - 9x + 28 = 0$.
8. $x^3 - 9x + 280 = 0$.
9. $8x^3 - 12x + 9 = 0$.
10. $4x^3 - 9x - 14 = 0$.
11. $x^3 + 4x - 5 = 0$.
12. $x^3 + 2x + 6 = 0$.

60.* Solution of quadratics by means of cubic parabolas.



To solve the quadratic

$$x^2 + px + q = 0 \qquad (1)$$

by means of a cubic parabola, multiply by $x - p$, *i.e.* introduce the new root p, $x^3 + (q - p^2)x - pq = 0$. (2)

Or if $\qquad y = x^3, \quad (3)$

$$y + (q - p^2)x - pq = 0. \qquad (4)$$

The line (4) passes through (p, p^3) and $(0, pq)$.

Thus, to solve $\qquad x^2 + 4x + 3 = 0,$

take in the cubic parabola a point P whose abscissa $= p = 4$, and on OY lay off $OB = pq = 12$.

The line PB determines the roots (P' and P'').

$$x = -1 \text{ or } -3.$$

NOTE. For examples see Exercise 9.

61. Complete cubic equations. To determine a method for the graphic solution of complete cubic equations, consider first a concrete example.

To solve $x^3 + 9x^2 + 20x + 12 = 0$. (1)

Substitute $x = z - (\frac{1}{3} \times \text{second coefficient})$.

Or $x = z - 3$. (2)

Then $(z-3)^3 + 9(z-3)^2 + 20(z-3) + 12 = 0$. (3)

If (3) were simplified, it would not contain the second power of z, for the first term produces $-9z^2$, the second term $+9z^2$, and the other terms do not contain z^2.

Hence equation (3) can be solved by one of the methods for incomplete cubic equations, but the one given in § 55 is the more convenient, since it does not require the simplification of the equation.

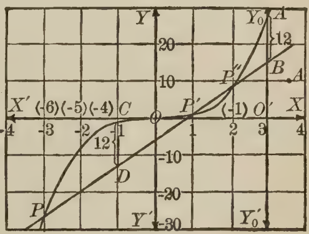
If $z = 3, z - 3 = 0,$

and $(z-3)^3 + 9(z-3)^2 + 20(z-3) + 12 = 12.$

If $z = -1, z - 3 = -4,$

and $(z-3)^3 + 9(z-3)^2 + 20(z-3) + 12 = 12.$

Consider z as abscissa, y as ordinate, and construct the cubic parabola $y = z^3$.



Through (3, 0) and $(-1, 0)$ draw ordinates and let them meet the curve in A and C. On the ordinates lay off downward $AB = 12$, and $CD = 12$, and draw BD.

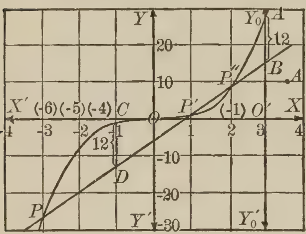
By measuring the abscissas of the points of intersection, we obtain the roots:

$$z = 2, \quad 1, \text{ and } -3.$$

Hence $x = -1, -2, \text{ and } -6.$

62. In the preceding diagram z represents the abscissas, but by changing the location of the y-axis we can obtain abscissas which equal x.

On OX lay off $OO' = 3$. Consider O' as the new origin, and the ordinate Y_0Y_0', drawn through O', as the new y-axis. Then the abscissa of any point is smaller by 3 than the old abscissa z, or the new abscissa is $z-3$, *i.e.* x. By thus introducing a new axis, the entire work of the preceding paragraph can be done without introducing z at all.



Thus, instead of saying:

If $\quad z-3=-4,\ (z-3)^3+9\,(z-3)^2+20\,(z-3)+12=12,$

we have briefly:

If $\quad x=-4,\ x^3+9\,x^2+20x+12=12.$

Similarly, instead of measuring the z, we may directly measure the x, and thus obtain the roots of (1).

63. A change in the position of the axes is called **a transformation of coördinates.**

To solve
$$x^3+bx^2+cx+d=0, \tag{4}$$
we locate O', the new origin, at the point $\left(\frac{b}{3}, 0\right)$, and consider the ordinate through O' as the new y-axis. If z is the old abscissa, then the new abscissa $x = z - \frac{b}{3}$, and this value substituted in (4) produces an equation without z^2.

Similarly, to solve
$$ax^3+bx^2+cx+d=0,$$
make
$$OO' = \frac{b}{3a}.$$

64. The method for solving complete cubics, which was derived in the preceding paragraphs, may be summarized as follows:

To solve the complete cubic,

$$ax^3 + bx^2 + cx + d = 0,$$

divide by a:

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0.$$

Construct the standard cubic parabola, and *after it is constructed* change the origin to the point $\left(\frac{b}{3a}, 0\right)$.

Locate two points by the method of § 55. The line which joins these points intersects the cubic parabola in one or more points whose abscissas are the required roots.

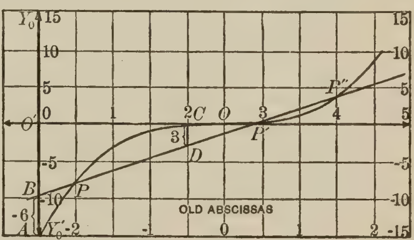
NOTE. In finding *real roots*, all work except the construction of the cubic parabola refers to the new y-axis, and the old axis may be omitted.

Ex. 1. Solve $2x^3 - 15x^2 + 31x - 12 = 0$.

Dividing by 2, and denoting the left member by y, we have

$$y = \frac{2x^3 - 15x^2 + 31x - 12}{2} = 0.$$

After drawing the standard cubic parabola (*i.e.* $y = z^3$), lay off on the x-axis $OO' = \frac{1}{3}\left(-\frac{15}{2}\right)$, *i.e.* $-2\frac{1}{2}$, and consider O' as the new origin.



If $x = 0, y = -6$.
If $x = 2, y = 3$.

Let the new y-axis (*i.e.* Y_0Y_0') meet the cubic parabola in A, and the ordinate through (2, 0) meet the curve in C. On AO' lay off upward $AB = 6$, and on the ordinate through C lay off downward $CD = 3$. Draw BD and measure the abscissas of the points of intersection P, P', and P''. Thus we obtain:

$$x = \tfrac{1}{2}, 3, \text{ and } 4.$$

65. Complex roots of complete cubics are determined by applying §§ 59 and 64. In using § 59 we find the ordinate

through the point of contact by producing the line PF from P to the y-axis by one half its own length. The student should bear in mind that this *refers to the old y-axis*, or that F lies in YY'.

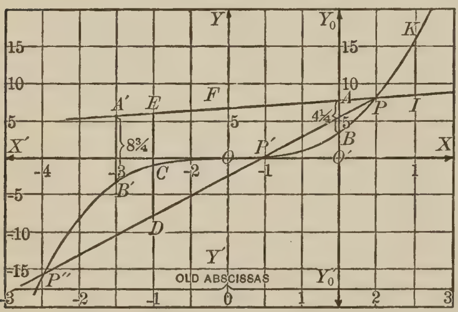
Ex. 2. Solve $4x^3 + 18x^2 + 24x - 17 = 0$.

Dividing by 4, $\quad y = \dfrac{4x^3 + 18x^2 + 24x - 17}{4} = 0.$

Construct the cubic parabola, lay off on the x-axis $OO' = \frac{1}{3}$ of $\frac{18}{4}$, *i.e.* $1\frac{1}{2}$, and consider O' the new origin.

If $\quad x = 0, \quad y = -4\frac{1}{4}$.

If $\quad x = -3, \ y = -8\frac{3}{4}$.



Locate the points A and A' in the usual manner ($AB = -4\frac{1}{4}$, $A'B' = -8\frac{3}{4}$), and draw AA', which meets the cubic parabola in P and the old y-axis in F. Produce PF by one half its own length to E, and let the ordinate through E meet the curve in C. Produce EC by its own length to D, and draw PD meeting the cubic parabola in P' and P''.

The abscissa of D is $-\frac{5}{2}$, and the difference of the abscissas of P' and D is $\frac{3}{2}$. Hence the required roots are

$$-\tfrac{5}{2} + \tfrac{3}{2}\sqrt{-1},\ -\tfrac{5}{2} - \tfrac{3}{2}\sqrt{-1}, \text{ and } \tfrac{1}{2}.$$

EXERCISE 17

Find graphically the real roots of the following equations: *

1. $x^3 - 3x^2 - x + 3 = 0$.
2. $x^3 - 9x^2 + 23x - 15 = 0$.
3. $x^3 - 6x^2 + 3x + 10 = 0$.
4. $x^3 - 8x^2 + 17x - 10 = 0$.

* For most of the following examples, a graph of the cubic parabola from $x = -3\frac{1}{2}$ to $x = 3\frac{1}{2}$ is sufficient. In other cases, apply the method of § 51.

5. $x^3+7x^2+14x+8=0$.

6. $x^3-2x^2-5x+6=0$.

7. $x^3-2x^2-4x+2=0$.

8. $x^3-3x^2+x+7=0$.

9. $x^3+4x^2-2x-5=0$.

10. $x^3+x^2+x+5=0$.

11. $2x^3+8x^2+2x-3=0$.

12. $5x^3-3x^2-20x+12=0$.

13. $2x^3-4x^2-10x+9=0$.

14. $2x^3-5x^2-4x+3=0$.

15. $4x^3-12x^2-19x+12=0$.

16. $4x^3-12x^2-31x+18=0$.

17. $x^3+6x^2-24x+60=0$.

Find the real and complex roots of the following equations:

18. $x^3-3x^2+x+5=0$.

19. $x^3+6x^2+10x+8=0$.

20. $x^3-3x^2+2x+6=0$.

21. $x^3+6x^2+13x+20=0$.

22. $x^3-6x^2+11x-12=0$.

23. $x^3+x^2-2x+12=0$.

24. $x^3+x^2-7x+15=0$.

25. $x^3-9x^2+28x-20=0$.

66. Values of a complete cubic function. The method for finding the values of a function for various values of x, as given in § 52, is true also for the complete cubic equation.

Thus, in the example of § 65:

If $x=-3$, $4x^3+18x^2+24x-17=4(A'B')=-35$.

If $x=1$, $4x^3+18x^2+24x-17=4(IK)=29$, etc.

NOTE. In order to make the new y-axis coincide with one of the lines of the cross-section paper, it is sometimes advisable to take the unit of the abscissas equal to the length of three squares of the paper.

67. Construction of the graph of a complete cubic function.

Ex. 3. Construct the graph of

$$y=x^3+4x^2-x-4.$$

$$OO'=\tfrac{1}{3}\cdot 4=\tfrac{4}{3}.$$

Take the unit of abscissas equal to the length of three squares (Note, § 66).

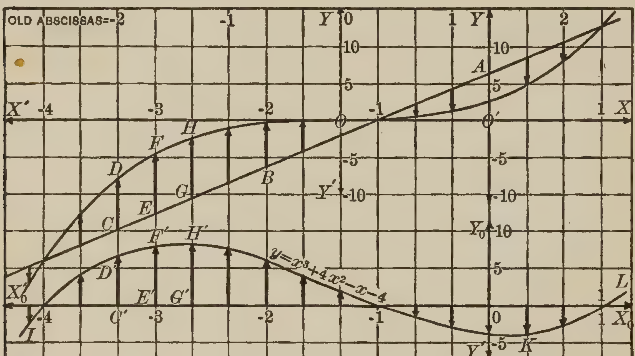
Construct the cubic parabola, and place the new origin at the point O'.

If $$x = 0,\ y = -4.$$

If $$x = -2,\ y = 6.$$

Locate the points A and B in the usual manner, and draw AB. Draw a new x-axis X_0X_0', and make $C'D' = CD$, $E'F' = EF$, $G'H' = GH$, etc.

By joining the points D', F', H', etc., in succession the required graph $IF'KL$ is obtained.



68. If the coefficient of x^3 is a, the student should keep in mind that in applying the above method every ordinate has to be multiplied by a.

EXERCISE 18

1. If $y = x^3 - 4x^2 + 2x + 5$, determine graphically the value of y if

$$(a)\ x = \tfrac{1}{3},\ (b)\ x = 1\tfrac{2}{3},\ (c)\ x = 2.$$

Construct, by means of the standard curve, the graphs of the following functions in rectangular coördinates:

2. $y = x^3 + 2x^2 - 5x - 7.$
3. $y = x^3 + 5x^2 - 3.$
4. $y = x^3 + 4x^2 + x + 2.$
5. $y = 4x^3 - 12x^2 - 19x + 12.$
6. $y = x^3 + 6x^2 - x - 30.$
7. $y = x^3 + x^2 - x + 15.$
8. $y = x^3 - 3x^2 + 7x + 5.$
9. $y = x^4 + 6x^2 + 2x - 9.$

Note. The solution of a cubic equation by means of a parabola or a rectangular hyperbola is given on §§ 75 and 84.

CHAPTER VII

BIQUADRATIC EQUATIONS

69. Solution of biquadratics in which the second term is wanting. To solve an incomplete biquadratic of the form

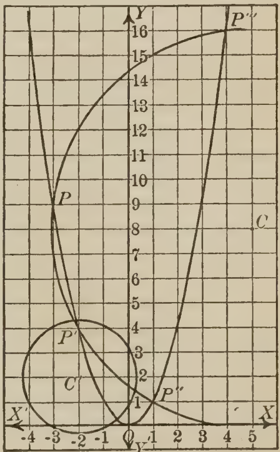
$$x^4 + bx^2 + cx + d = 0, \tag{1}$$

write this equation as follows:

$$x^4 + (b-1)x^2 + x^2 + cx + d = 0.$$

$$\left.\begin{array}{lr} \text{Let} & y = x^2. \\ \text{Then} & y^2 + (b-1)y + x^2 + cx + d = 0. \end{array}\right\} \begin{array}{r}(2)\\(3)\end{array}$$

The solution of the system (2), (3) for x produces the required roots. But the graph of (2) is a parabola which is identical for all biquadratic equations, while the graph of (3) is a circle (§ 27).



Ex. 1. Solve $x^4 - 15x^2 - 10x + 24 = 0$. (1)

Separate $-15x^2$ into two parts, one of which is x^2:

$$x^4 - 16x^2 + x^2 - 10x + 24 = 0.$$

$$\left.\begin{array}{lr} \text{Let} & y = x^2. \\ \text{Then} & y^2 - 16y + x^2 - 10x + 24 = 0. \end{array}\right\} \begin{array}{r}(2)\\(3)\end{array}$$

To construct (3) transpose 24 and complete the squares,

$$y^2 - 16y + 64 + x^2 - 10x + 25$$
$$= -24 + 64 + 25.$$

Or $(y-8)^2 + (x-5)^2 = (\sqrt{65})^2$. (3)

I.e. (3) is a circle whose center C is the point $(5, 8)$ and whose radius equals $\sqrt{65}$.*

* $\sqrt{65} = \sqrt{8^2 + 1^2}$, hence the line joining C and $(4, 0)$ is the radius. In other cases, use table of square roots, Appendix III.

Equation (2) is the standard parabola, which is intersected by the circle in four points, P, P', P'', and P'''. The abscissas of P, P', P'', and P''' are the required roots. $\therefore x = -3, -2, 1$, or 4.

NOTE. The student should remember that in problems involving circles, the same scale unit must be used for abscissas and ordinates.

70. Formulæ for radius and origin. According to the preceding paragraph, the equation

$$x^4 + bx^2 + cx + d = 0 \tag{1}$$

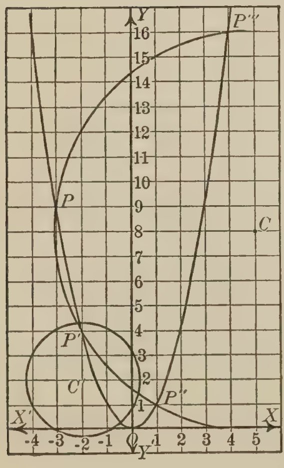
is solved by the system

$$\left.\begin{aligned} y &= x^2, \qquad &(2)\\ x^2 + cx + y^2 + (b-1)\,y + d &= 0. \qquad &(3)\end{aligned}\right\}$$

Transposing and completing the squares in (3),

$$x^2 + cx + \left(\frac{c}{2}\right)^2 + y^2 + (b-1)\,y + \left(\frac{b-1}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{b-1}{2}\right)^2 - d.$$

Or
$$\left(x + \frac{c}{2}\right)^2 + \left(y + \frac{b-1}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{b-1}{2}\right)^2 - d.$$



If we denote the coördinates of the center of the circle by x_0 and y_0, and the radius by r, we have

$$x_0 = -\frac{c}{2}, \tag{4}$$

$$y_0 = \frac{1-b}{2}, \tag{5}$$

$$r^2 = x_0^2 + y_0^2 - d. \tag{6}$$

Ex. 2. Solve by means of the formulæ:

$$x^4 - 3x^2 + 4x + 3 = 0.$$

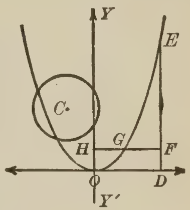
$$\begin{aligned} x_0 &= -2,\\ y_0 &= 2,\\ r^2 &= 5.\end{aligned}$$

I.e. the center C' of the circle is $(-2, 2)$, and its radius is $\sqrt{5}$, or the line that joins C' and $(-1, 0)$.

There are only two points of intersection, and hence two roots are real, and two complex.

The real roots are $-.6$ and -2.1.

71.* The expression $\sqrt{x^2+y^2-d}$ can be constructed geometrically. If C is the center of the circle, lay off on the x-axis $OD = OC$, and draw the ordinate DE, which equals x^2+y^2. On ED lay off $EF = d$ and draw $FH \parallel XO$. The segment HG intercepted on this parallel by the y-axis and the parabola, equals r.



EXERCISE 19

Find the real roots of the following equations: *

1. $x^4+5x^2+4x-28=0$.
2. $x^4-15x^2+10x+24=0$.
3. $x^4-x^2+4x-4=0$.
4. $x^4-19x^2+2x+56=0$.
5. $x^4-5x^2+4=0$.
6. $x^4-7x^2-12x+18=0$.
7. $x^4-4x^2+12x+9=0$.
8. $x^4-7x^2-6x+12=0$.
9. $x^4-9x^2-2x+6=0$.
10. $x^4+4x^2-5x-55=0$.
11. $x^4-6x^2+3x+2=0$.
12. $x^4-15x^2-10x+24=0$.

72. Solution for large roots. To use the same diagram of the standard curve for the finding of large and small roots of the equation

$$x^4+bx^2+cx+d=0, \tag{1}$$

multiply the values of the abscissas and ordinates in the diagram by any number, as p. Then the equation of the parabola becomes

$$py=x^2. \tag{2}$$

Equation (1) may be written in the form

$$x^4+(b-p^2)x^2+p^2x^2+cx+d=0.$$

Partly substituting py for x^2,

$$p^2y^2+(b-p^2)py+p^2x^2+cx+d=0.$$

* For the following exercises a graph from $x=-4$ to $x=+4$ is sufficient.

The last equation is easily transformed into the following one (§ 27):

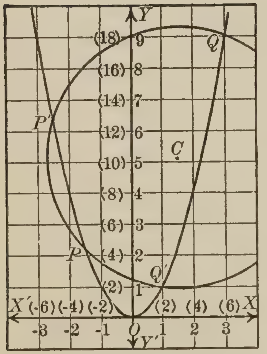
$$\left(x+\frac{c}{2p^2}\right)^2+\left(y+\frac{b-p^2}{2p}\right)^2=\left(\frac{c}{2p^2}\right)^2+\left(\frac{b-p^2}{2p}\right)^2-\frac{d}{p^2}. \quad (3)$$

Equation (3) represents a circle whose center and radius are determined by the formulæ

$$x_0=-\frac{c}{2p^2}, \quad (4)$$

$$y_0=\frac{p^2-b}{2p}, \quad (5)$$

$$r^2=x_0^{\,2}+y_0^{\,2}-\frac{d}{p^2}. \quad (6)$$



The abscissas of the points of intersection of the circle (3) and the parabola (2) are the required roots.

Ex. Solve

$$x^4-37x^2-24x+180=0.$$

Since obviously x_0 and y_0 are very large, multiply the values on the two axes by 2; *i.e.* make $p=2$. (The new values are given in parentheses.)

Applying formulæ (4), (5), and (6), we have:

$$x_0=3,$$
$$y_0=10\tfrac{1}{4},$$
$$r=8.3+.*$$

Construct the circle and measure the abscissas of the points of intersection. Hence

$$x=-5, -3, 2, 6.$$

EXERCISE 20

Solve graphically:

1. $x^4-45x^2-40x+84=0$.
2. $x^4-42x^2-64x+105=0$.
3. $x^4-23x^2-18x+40=0$.
4. $x^4-37x^2-24x+180=0$.

* To compute r, use table of squares and square roots in Appendix III.

5. $x^4 - 75x^2 - 70x + 144 = 0.$
6. $x^4 - 63x^2 + 50x + 336 = 0.$
7. $x^4 - 55x^2 - 30x + 504 = 0.$
8. $x^4 - 58x^2 + 441 = 0.$
9. $x^4 - 49x^2 + 36x + 252 = 0.$
10. $x^4 - 49x^2 - 36x + 252 = 0.$

73. Complex roots. If an equation has two real and two complex roots, the roots may be found by a method similar to the one employed for quadratics and cubics (§§ 41 and 59).

Let the equation $x^2 + px + q = 0$ have equal roots, and d be a positive quantity. Consider the following three equations which are supposed not to contain x^3 when simplified:

$$(x-a)(x-b)(x^2+px+q+d)=0, \tag{1}$$

$$(x-a)(x-b)(x^2+px+q)=0, \tag{2}$$

$$(x-a)(x-b)(x^2+px+q-d)=0. \tag{3}$$

Then the roots are (§ 58) respectively:

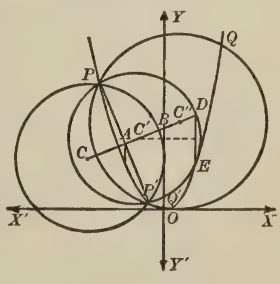
$$a,\ b,\ -\frac{p}{2} \pm \sqrt{-d};$$

$$a,\ b,\ -\frac{p}{2},\ -\frac{p}{2};$$

$$a,\ b,\ -\frac{p}{2} \pm \sqrt{d}.$$

I.e. the roots of equation (1) are complex, but they may be found by solving (3) instead.

The circles C, C', and C'', which represent respectively equations (1), (2), and (3), are connected by simple geometric relations, which make it possible to construct the third circle (C'') when the first one (C) is given.



1. *The three circles C, C', and C'', pass through two points, P and P', in the parabola, the abscissas of P and P' being a and b.*

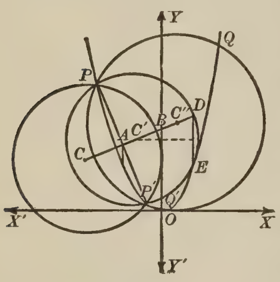
Obviously a and b are roots of the equations (1), (2), and (3).

2. *The three centers C, C', and C'', lie in the perpendicular bisector of PP'.*

3. *The second center, C', bisects the line joining the other two, C and C'', or $CC' = C'C''$.*

If the ordinate of C' is m, then the ordinate of C is $m - \frac{d}{2}$, for the coefficient of x^2 in (1) is greater by d than the coefficient of x^2 in (2) (§ 70). Similarly, the ordinate of C'' equals $m + \frac{d}{2}$.

74. Hence if circle C is given and it intersects the parabola in P and P', construct AB, the perpendicular bisector of PP', and in AB determine C', the center of the circle that passes through P and P', and touches the parabola in another point, E. Produce CC' by its own length to C'', and from C'', with a radius equal to $C''P$, draw a circle. This circle intersects the parabola in two other points, Q and Q'. If the abscissas of Q and Q' are $m + n$ and $m - n$, the required roots are respectively $m + n\sqrt{-1}$ and $m - n\sqrt{-1}$.



The abscissa of E is always equal to m, and the difference of the abscissas of Q and E (or E and Q') is equal to n.

A convenient method for constructing the circle C', which touches the parabola and passes through P and P', is the following:

Let the perpendicular bisector of PP' meet PP' in A, and the y-axis in B. Produce AB by its own length to D, and let the ordinate through D meet the parabola in E. Then E is the point of contact, and the perpendicular bisector of DE meets CD in the required point C'.

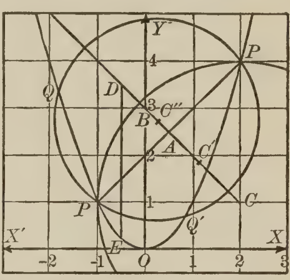
Ex. Solve the equation

$$x^4 - x^2 - 4x - 4 = 0.$$

According to § 70,

$$x_0 = 2,\ y_0 = 1,\ r = 3.$$

The circle drawn from (2, 1), or C, as center with a radius equal to 3 intersects the parabola in only two points, P and P'. Hence there are only two real roots, viz. -1 and 2.



Draw AB, the perpendicular bisector of PP', and let it meet the y-axis in B. Produce AB by its own length to D, and draw the ordinate DE, E being a point in the parabola. The perpendicular bisector of DE meets AB in C', the center of the second circle. Produce CC' by its own length to C'' and from C'' as a center draw a circle through P and P'. This circle, C'', meets the parabola in two other points, Q and Q'.

The abscissa of E, *i.e.* $-\frac{1}{2}$, is the real part, and the difference of the abscissas of E and Q (or E and Q'), *i.e.* 1.3, is the imaginary part of the required roots.

Hence the roots are:

$$-\tfrac{1}{2} \pm 1.3\sqrt{-1},\ 2, \text{ and } -1.$$

EXERCISE 21

Find the real and complex roots of the following equations:

1. $x^4 - 8x^2 + 8x + 15 = 0.$
2. $x^4 - 7x^2 + 12x + 18 = 0.$
3. $x^4 - 4x^2 - 12x - 9 = 0.$
4. $x^4 - 5x^2 - 10x - 6 = 0.$
5. $x^4 - 5x^2 - 4x + 12 = 0.$
6. $x^4 + 3x^2 + 6x - 10 = 0.$
7. $x^4 - 11x^2 - 14x + 24 = 0.$
8. $x^4 - 10x^2 + 20x - 16 = 0.$
9. $x^4 - 5x^2 + 10x - 6 = 0.$
10. $x^4 - 8x^2 + 8x + 15 = 0.$

75.* Solution of cubic equations by means of the standard parabola.

To solve the cubic

$$x^3 + bx^2 + cx + d = 0 \tag{1}$$

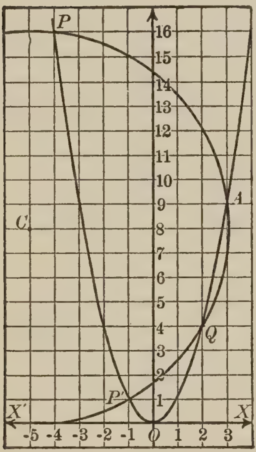
by means of a parabola, multiply by $(x - b)$, *i.e.* introduce the new root b.

$$x^4 + (c - b^2)x^2 + (d - bc)x - bd = 0.$$

Hence, applying § 70, we have

$$x_0 = \frac{bc - d}{2}, \tag{2}$$

$$y_0 = \frac{b^2 - c + 1}{2}. \tag{3}$$



The formula for the radius is not necessary, since the circumference must pass through the point of the parabola whose abscissa is b, *i.e.* (b, b^2).

Thus, to solve

$$x^3 + 3x^2 - 6x - 8 = 0, \tag{4}$$

either multiply by $x - 3$, obtaining $x^4 - 15x^2 + 10x + 24 = 0$, or apply directly formulæ (2) and (3).

Hence $x_0 = -5$,

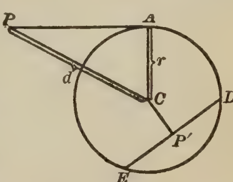
$y_0 = 8$.

From $(-5, 8)$ as a center construct a circle passing through A, *i.e.* the point in the parabola whose abscissa is 3.

Hence the roots of (4) are -4, -1, 2.

[For examples see Exercise 16.]

76. Power of a point with respect to a circle. If r is the radius of a circle C, and d the distance of a point P from its center, $d^2 - r^2$ is called the *power of the point P with respect to circle C.*



If P lies *without* the circle the power is *positive* and equal to the square of the tangent drawn from P to the circle.*

If the point lies *within* the circle, as P', the power is *negative.* If a chord DE is drawn perpendicular to CP', the power of P' is equal to $-(P'D)^2$.

77. Values of a biquadratic function. To solve

$$x^4 + bx^2 + cx + d = 0, \tag{1}$$

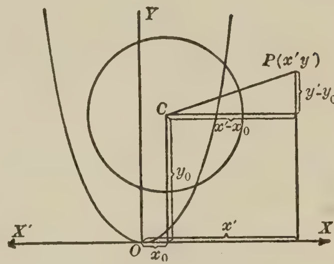
we substitute

$$y = x^2, \tag{2}$$

* Schultze and Sevenoak's Geometry, § 311.

and obtain the equation of a circle (§ 70),

$$(x - x_0)^2 + (y - y_0)^2 - r^2 = 0. \tag{3}$$



If any point P, whose coördinates are x' and y', is joined to (x_0, y_0) *i.e.* C, we have (Geometry, § 310)

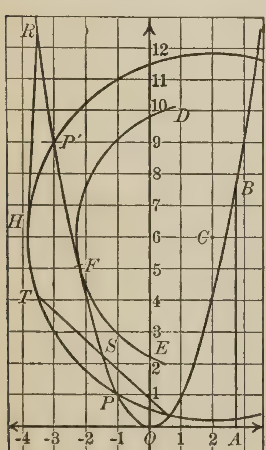
$$\overline{PC}^2 = (x' - x_0)^2 + (y' - y_0)^2.$$

Hence $\overline{PC}^2 - r^2$

$$= (x' - x_0)^2 + (y' - y_0)^2 - r^2.$$

I.e. if we substitute the coördinates of any point P in the left member of (3), this member becomes equal to the power of P with reference to circle (3).

If the point P is located in the parabola, then $y' = x'^2$ and the left member of (3) becomes equal to the left member of (1). Hence, *the value of the function* (1) *for any particular value* x' *is equal to the power of point* (x', x'^2) *with respect to circle* (3).



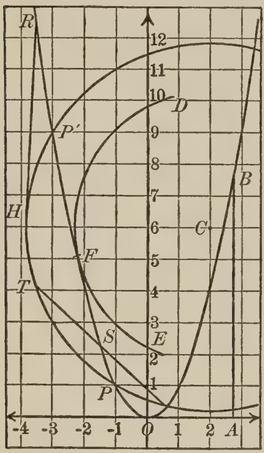
78. Thus, to find the various values of the function $y = x^4 - 11x^2 - 4x + 6$, construct a circle so that

$$x_0 = 2,\ y_0 = 6,\ r = \sqrt{34}. \quad (\S\ 70)$$

To find y if $x = -3\frac{1}{2}$, locate in the parabola a point R, whose abscissa is $-3\frac{1}{2}$, and draw the tangent RH to circle C. The required value equals $(RH)^2 = (6^-)^2 = 36^-$.

Similarly, if $x = -1\frac{1}{2}$, locate in the parabola a point S whose abscissa equals $-1\frac{1}{2}$, and draw $ST \perp CS$, then $y = -(ST)^2$. To find $(ST)^2$ graphically, make $OA = ST$, then the ordinate $AB = (ST)^2$, or the function equals $-AB = -7.7$.

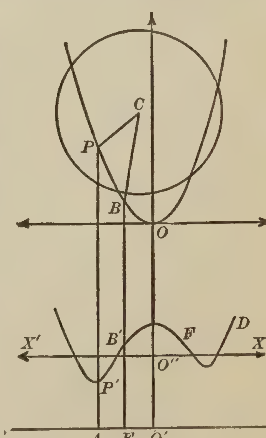
Many other problems relating to the value of the functions may be solved by such a diagram. Thus, to find which value of x between $x = -4$ and $x = 0$ produces the smallest value of y, determine in the parabola the point nearest to C by drawing an arc EFD from C, touching the parabola in F. The abscissa of F, *i.e.* -2.2, is the required value.



Similarly, we can determine the greatest value of the function, the value of x, if the function is given, etc.

79. The graph of a biquadratic function in rectangular coördinates can be constructed by means of § 77. Since $y = \overline{PC}^2 - r^2$, construct first the curve whose ordinates are $\overline{PC}^2$; *i.e.* make $AP' = \overline{CP}^2$, $EB' = \overline{CB}^2$, etc. (Use table in Appendix III, and draw new ordinates on smaller scale.)

Locate in this manner a sufficient number of points, P', B', F, D, etc., and draw the curve $P'B'FD$. Make $O'O'' = r^2$ and through O'' draw $XX' \perp O'O''$. Then the curve $P'B'CD$ referred to O'' as origin is the required graph.



EXERCISE 22

If $y = x^4 - 3x^2 + 4x + 3$, find graphically the value of y,

1. If $x = 2$.
2. If $x = 3$.
3. If $x = 4$.
4. If $x = 3.2$.
5. If $x = 2.7$.
6. Construct the graph of y in rectangular coördinates.

80. Complete biquadratic equations. The complete biquadratic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0 \qquad (1)$$

is transformed into another equation without the cubic term by the substitution

$$x = z - \frac{a}{4}.$$

The resulting equation can be solved, and by subtracting $\frac{a}{4}$ from the answers the roots of (1) are obtained.

Ex. Solve

$$x^4 + 4x^3 - 5x^2 - 22x - 8 = 0. \qquad (1)$$

Substituting $\quad x = z - \frac{4}{4} = z - 1, \qquad (2)$

$(z-1)^4 + 4(z-1)^3 - 5(z-1)^2 - 22(z-1) - 8 = 0.$

Simplifying,

$$z^4 - 11z^2 - 4z + 6 = 0.*$$

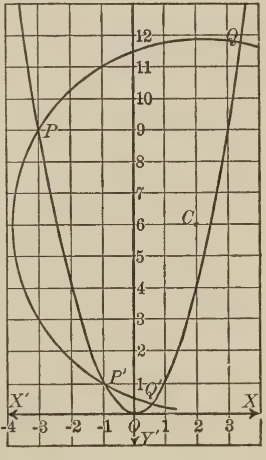
$$\therefore \quad z_0 = 2, \; y_0 = 6, \; r = \sqrt{34}.$$

Drawing the circle and measuring the abscissas of the points of intersection, we obtain

$$z = -3, \; -1, \; .6, \; 3.4.$$

Hence, from (2),

$$x = -4, \; -2, \; -.4, \; 2.4.$$



81. In most cases, the method of the preceding exercise is the best. It is possible, however, to derive general formulæ.

If $\qquad x^4 + ax^3 + bx^2 + cx + d = 0, \qquad (1)$

let $\qquad x = z + p$, where $p = -\frac{a}{4}. \qquad (2)$

* Students who are familiar with the general method for removing the second term should of course use this method. See Schultze's Advanced Algebra, § 566.

Then equation (1) becomes

$$y^4+(6\,p^2+3\,ap+b)z^2+(4\,p^3+3\,ap^2+2\,bp+c)z$$
$$+p^4+ap^3+bp^2+cp+d=0. \qquad (3)$$

Considering that $p=-\frac{a}{4}$, we can easily obtain the following values:

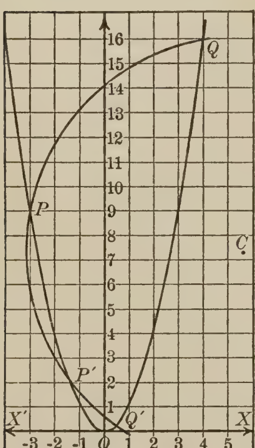
$$p=-\frac{a}{4},$$

$$z_0=\frac{-2\,ap^2-2\,bp-c}{2},$$

$$y_0=\frac{2-3\,ap-2\,b}{4},$$

$$r^2=x_0^{\,2}+y_0^{\,2}-(p^4+ap^3+bp^2+cp+d).*$$

Constructing the circle (z_0, y_0, r) and measuring the abscissas of the points of intersection, produces the roots of (3), and hence those of (1).



Thus, in the preceding equation,

$$x^4+4\,x^3-5\,x^2-22\,x-8=0,$$

we obtain

$$p=-1.$$

$$z_0=\frac{-8-10+22}{2}=2.$$

$$y_0=\frac{2+12+10}{4}=6.$$

$$r^2=4+36-(1-4-5+22-8)=34.$$

Ex. Solve $4\,x^4+16\,x^3-31\,x^2-139\,x-60=0.$

Dividing by 4, $x^4+4\,x^3-\frac{31}{4}x^2-\frac{139}{4}x-15=0.$

* Students familiar with Calculus can, by means of Taylor's series, obtain

$$z_0=-\frac{f'(p)}{2},\ y_0=\frac{2-f''(p)}{4},\ r^2=x_0^{\,2}+y_0^{\,2}-f(p).$$

Hence $p = -1,$

$$z_0 = 5\tfrac{5}{8},$$
$$y_0 = 7\tfrac{3}{8},$$
$$r = 8.8.$$

The construction of the circle produces the values $z = -3, -1\frac{1}{2}, \frac{1}{2}, 4.$

Hence $x = z + p = z - 1.$

Or $x = -4, -2\frac{1}{2}, -\frac{1}{2}, 3.$

EXERCISE 23

Solve the equations:

1. $x^4 + 4x^3 - 9x^2 - 16x + 20 = 0.$
2. $x^4 - 4x^3 - 17x^2 + 24x + 36 = 0.$
3. $x^4 - 8x^3 + x^2 + 78x - 72 = 0.$
4. $x^4 + 8x^3 + 14x^2 - 8x - 15 = 0.$
5. $x^4 + 4x^3 - 4x^2 - 16x = 0.$
6. $x^4 - 2x^3 - 16x^2 + 2x + 15 = 0.$
7. $x^4 + 4x^3 - 21x^2 - 64x + 80 = 0.$
8. $x^4 + 8x^3 - 3x^2 - 62x + 56 = 0.$
9. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
10. $x^4 + 3x^3 - 8x^2 - 12x + 16 = 0.$
11. $x^4 - 4x^3 - 5x^2 + 22x - 8 = 0.$

82.* Solution of biquadratics by means of the hyperbola $y = \dfrac{1}{x}$.

Let us first consider the equation

$$x^4 + ax^3 + bx^2 + cx + 1 = 0. \qquad (1)$$

Partly replacing x by $\dfrac{1}{y}$,

$$\frac{x^2}{y^2} + \frac{ax}{y^2} + \frac{b}{y^2} + \frac{c}{y} + 1 = 0.$$

Or $x^2 + ax + b + cy + y^2 = 0.$

Applying § 27,

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{c}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - b. \qquad (3)$$

I.e. the required roots are determined by the points of intersection of the standard curve $y = \dfrac{1}{x}$ (2) and the circle (3).

The circle is determined by the formulæ

$$x_0 = -\frac{a}{2},\ y_0 = -\frac{c}{2},\ r^2 = x_0^2 + y_0^2 - b.$$

83.* The equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

may be solved by the circle :

$$x_0 = -\frac{a}{2d^{\frac{1}{4}}},\ y_0 = -\frac{c}{2d^{\frac{3}{4}}},\ r^2 = x_0^2 + y_0^2 - \frac{b}{d^{\frac{1}{2}}}.$$

The abscissas of the points of intersection multiplied by $d^{\frac{1}{4}}$ are the required roots.

84.* Solution of a cubic by the hyperbola $\mathbf{y = \frac{1}{x}}$.

To solve $$x^3 + bx^2 + cx + d = 0, \qquad (1)$$

multiply by $x + \frac{1}{d}$, *i.e.* introduce the new root $-\frac{1}{d}$.

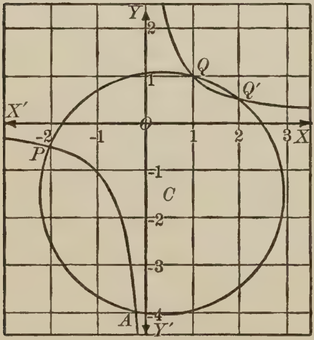
$$x^4 + \left(b + \frac{1}{d}\right)x^3 + \left(c + \frac{b}{d}\right)x^2 + \left(d + \frac{c}{d}\right)x + 1 = 0.$$

Hence, by § 82, we have to construct the circle that is determined by the formulæ :

$$x_0 = -\frac{1}{2}\left(b + \frac{1}{d}\right),$$

$$y_0 = -\frac{1}{2}\left(d + \frac{c}{d}\right).$$

The formula for the radius is not necessary, since the circle must pass through the point $\left(-\frac{1}{d}, -d\right)$.



Ex. Solve $x^3 - x^2 - 4x + 4 = 0$.

$$x_0 = -\tfrac{1}{2}(-1 + \tfrac{1}{4}) = \tfrac{3}{8},$$

$$y_0 = -\tfrac{1}{2}(\ \ 4 - 1) = -\tfrac{3}{2}.$$

From (x_0, y_0) as center draw a circle through $(-\frac{1}{4}, -4)$, *i.e.* A. By measuring the abscissas of the other points of intersection, we obtain

$$x = -2, 1, 2.$$

NOTE. The preceding construction can be used advantageously for large roots, since the ordinates do not become as large as in the case of the cubic parabola.

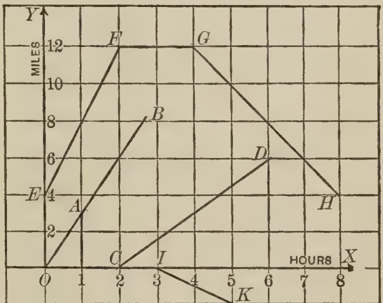
APPENDIX

I. GRAPHIC SOLUTION OF PROBLEMS

85. Problems are usually solved in algebra by expressing the conditions of the problems in the form of equations. By using the graphic method, however, many problems can be solved directly, without obtaining equations.

The fact that the graph of two proportional variables is a straight line is often useful. Thus, if x and y are the coördinates of a point, the following variables are represented by straight lines: $x =$ time, $y =$ distance covered by body moving uniformly; $x =$ time, $y =$ work done by a person; $x =$ volume, $y =$ weight of a body; $x =$ time, $y =$ quantity of water flowing through a pipe at a uniform rate, etc.

86. Uniform motion. To represent graphically the motion of a person traveling three miles per hour, it is only necessary to locate one point, *e.g.* (1, 3) or A, and to connect this point to the origin.



The increase of the ordinate per hour equals the rate of travel, *i.e.* 3 miles per hour.

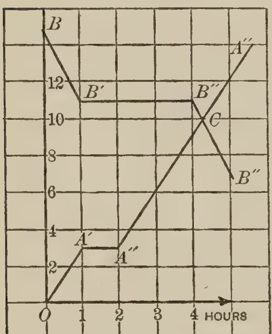
Similarly, CD represents the motion of another person who started two hours later and traveled $1\frac{1}{2}$ miles per hour.

$EFGH$ represents graphically that a third person had a start of 4 miles, traveled for 2 hours at the rate of 4 miles per

hour, then rested 2 hours, and finally returned to the starting point at the rate of 2 miles per hour.

IK represents graphically the motion of a fourth person who started 3 hours after the first and traveled in the opposite direction at the rate of 1 mile per hour.

Ex. 1. A and B start walking from two towns 15 miles apart, and walk toward each other. A walks at the rate of 3 miles per hour, but rests 1 hour on the way; B travels at the rate of 4 miles per hour and rests 3 hours. In how many hours do they meet?



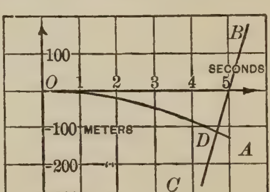
Construct the graphs $OA'A''A'''$ and $BB'B''B'''$. The abscissa of C, the point of intersection, is the required time.

Hence A and B meet in $4\frac{1}{4}$+ hours.

Ex. 2. A stone is dropped into a well, and the sound of its impact upon the water is heard at the top of the well 5 seconds later. If the velocity of sound is assumed as 360 meters per second, and $g = 10$ meters, how deep is the well? (A body falls in t seconds $\frac{g}{2}t^2$ meters.)

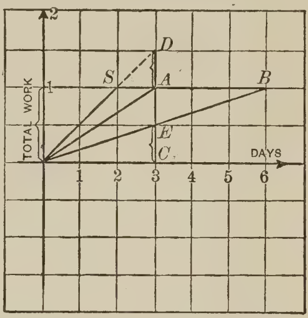
Construct the graph *ODA* of the falling body, making the distances negative, to indicate the downward motion. Since the motion of the sound is an upward motion, its graph *CB* is obtained by joining $(4, -360)$ and $(5, 0)$. The ordinate of the point of intersection *D* is the required number.

Hence depth of well = 110 meters.



87. Problems relating to work done, and to quantity of water flowing through a pipe, are quite similar to those of the preceding paragraph.

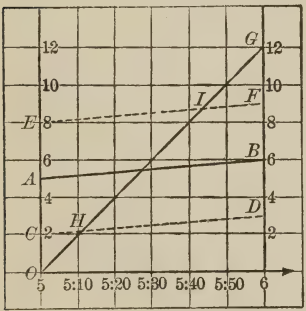
Ex. 3. A can do a piece of work in 3 days, and B in 6 days. In how many days can both do it, working together?



Make the hour equal to the unit of abscissas, and the work to be done equal to the unit of the ordinates. Then OA and OB represent the work done by A and B respectively. To obtain the graph of the work done by both together, we add the ordinates corresponding to any particular time, *e.g.* 3 hours; *i.e.* produce CA to D so that $AD = CE$. Then OD is the graph of the work done by both, and the required time is equal to abscissa of S, or two days.

Hence both working together will do the work in two days.

Ex. 4. At what time between 5 and 6 o'clock are the hands of a clock at right angles?



Let the abscissa represent the time from 5 to 6, and the ordinate the hour spaces.

It can easily be seen that AB represents the motion of the hour hand. A point 90° distant from the hour hand moves in the same time from 2 to 3 or from 8 to 9. Hence the motion of such a point is represented by CD or EF. But the graph of the motion of the minute hand is OG. Therefore the abscissas of the points H and I represent the required time. Or the hands are at right angles at 5 : 11 and 5 : $43\frac{1}{2}$.

EXERCISE 24

1. A sets out walking at the rate of 3 miles per hour, and 3 hours later B follows on horseback, traveling at the rate of 6 miles per hour. After how many hours will B overtake A, and how far will each then have traveled?

2. A and B set out walking at the same time in the same direction, but A has a start of 3 miles. If A walks at the rate of $2\frac{1}{2}$ miles per hour, and B at the rate of 3 miles per hour, how far must B walk before he overtakes A?

3. A train traveling 30 miles per hour starts $\frac{3}{4}$ of an hour before a second train that travels 35 miles an hour. In how many hours will the first train be overtaken by the second?

4. A sets out walking at the rate of 3 miles per hour, and one hour later B starts from the same point, traveling by coach in the opposite direction at the rate of 6 miles per hour. After how many hours will they be 27 miles apart?

5. A and B start walking at the same hour from two towns $17\frac{1}{2}$ miles apart, and walk toward each other. If A walks at the rate of 3 miles per hour and B at the rate of 4 miles per hour, after how many hours do they meet, and how many miles does A walk?

6. An accommodation train runs according to the following schedule:

Station	Distance from A	Arrives	Leaves
A	0		2
B	10	2:20	2:24
C	15	2:32	2:35
D	25	2:50	2:55
E	40	3:20	3:21
F	50	3:40	

An express train leaves A at 2:15 and reaches F at 3:25. Where does it overtake the accommodation train, if we assume that both trains move uniformly?

7. In how many days can A and B working together do a piece of work if each alone can do it in the following number of days?

(*a*)	A	in	6,	B	in	6.
(*b*)	A	in	12,	B	in	4.
(*c*)	A	in	12,	B	in	6.
(*d*)	A	in	20,	B	in	5.

8. A can do a piece of work in 18 days, B in 9, and C in 12 days. In how many days can all three do it working together?

9. At what time between 2 and 3 o'clock are the hands of the clock together?

10. At what time between 3 and 4 o'clock are the hands of a clock in a straight line and opposite?

11. At what time between 6 and 7 o'clock are the hands at right angles?

12. A cistern can be filled by two pipes in 3 and 6 hours respectively. In how many hours can it be filled by the two running together?

13. A cistern can be filled by pipes in 3, 4, and 5 hours respectively. In how many hours can it be filled by all three together?

14. A stone is dropped into a well and the sound of its impact upon the water is heard at the top of the well (*a*) 4 (*b*) 6 seconds later. If the velocity of sound is assumed as 360 meters per second, and $g = 10$ meters, how deep is the well? (A body falls in t seconds $\frac{g}{2}t^2$ meters.)

II. STATISTICAL DATA SUITABLE FOR GRAPHIC REPRESENTATION

1. Table of Population (in Millions) of United States, France, Germany, and British Isles

Year	U.S.	France	Germany	British Isles
1800	5.3	27.2	22.0	16.0
1810	7.2	28.8	23.4	17.6
1820	9.6	30.5	26.2	20.5
1830	12.9	32.4	29.7	24.0
1840	17.0	34.0	32.4	26.4
1850	23.2	35.6	35.2	27.2
1860	31.4	37.3	38.1	28.7
1870	38.6	36.1	40.5	31.2
1880	50.2	37.6	45.2	34.5
1890	62.6	38.6	49.4	37.5
1900	76.3	38.9	56.4	41.2

2. Arrival of Immigrants (in Ten Thousands), 1891–1905

From	'91	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04	'05
Germany	11	13	10	6	4	3	2	2	2	2	2	3	4	5	4
Italy	8	6	7	4	4	7	6	6	8	10	14	18	23	19	22
Russia	5	8	4	4	3	5	3	3	6	9	9	10	14	15	18

3. Population of New York City

1653	1,120	1756	10,530
1661	1,743	1771	21,865
1673	2,500	1774	22,861
1696	4,455	1786	23,688
1731	8,256	1790	33,131
1750	10,000	1800	60,489

3. POPULATION OF NEW YORK CITY — *Continued*

1805	75,587	1875	1,041,886
1810	96,373	1880	1,206,299
1816	100,619	1890	1,515,301
1820	123,706	1893	1,891,306
1825	166,136	1898 (all Boro's)	3,350,000
1830	202,589	1899 (all Boro's)	3,549,558
1835	253,028	1900 (all Boro's)	3,595,936
1840	312,710	1901 (all Boro's)	3,437,202
1845	358,310	1902 (all Boro's)	3,582,930
1850	515,547	1903 (all Boro's)	3,632,501
1855	629,904	1904 (all Boro's)	3,750,000
1860	813,669	1905 (all Boro's)	3,850,000
1865	726,836	1906 (all Boro's)	4,014,304
1870	942,292		

4. POPULATION (IN HUNDRED THOUSANDS) OF ILLINOIS, MASSACHUSETTS, NEW YORK, AND VIRGINIA

STATE	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900
Illinois			.5	1.6	4.8	8.5	17.1	25.3	30.8	38.3	48.2
Mass.	3.4	3.8	4.0	4.5	4.7	5.8	6.9	7.8	9.3	10.4	11.9
N. York	6.9	9.6	13.7	19.2	24.3	31.0	38.8	43.8	50.8	60.0	72.7
Virginia	8.8	9.7	10.7	12.1	12.4	14.2	16.0	12.3	15.1	16.6	18.5

5. TABLE OF MORTALITY

Completed Age	Number Surviving at Each Age	Deaths in Each Year	Number of Years Expectation	Number Dying annually out of Each 1000
10	100,000	749	48.7	7.49
11	99,251	746	48.1	7.52
12	98,505	743	47.4	7.54
13	97,762	740	46.8	7.57
14	97,022	737	46.2	7.60
15	96,285	735	45.5	7.63
16	95,550	732	44.9	7.66
17	94,818	729	44.2	7.69
18	94,089	727	43.5	7.73
19	93,362	725	42.9	7.77
20	92,637	723	42.2	7.81

5. Table of Mortality — *Continued*

Completed Age	Number Surviving at Each Age	Deaths in Each Year	Number of Years Expectation	Number Dying annually out of Each 1000
21	91,914	722	41.5	7.86
22	91,192	721	40.9	7.91
23	90,471	720	40.2	7.96
24	89,751	719	39.5	8.01
25	89,032	718	38.8	8.07
26	88,314	718	38.1	8.13
27	87,596	718	37.4	8.20
28	86,878	718	36.7	8.26
29	86,160	719	36.0	8.35
30	85,441	720	35.3	8.43
31	84,721	721	34.6	8.51
32	84,000	723	33.9	8.61
33	83,277	726	33.2	8.72
34	82,551	729	32.5	8.83
35	81,822	732	31.8	8.95
36	81,090	737	31.1	9.09
37	80,353	747	30.4	9.23
38	79,611	749	29.6	9.41
39	78,862	756	28.9	9.59
40	78,106	765	28.2	9.79
41	77,341	774	27.5	10.01
42	76,567	785	26.7	10.25
43	75,782	797	26.0	10.52
44	74,985	812	25.3	10.83
45	74,173	828	24.5	11.16
46	73,345	848	23.8	11.56
47	72,497	870	23.1	12.00
48	71,627	896	22.4	12.51
49	70,731	927	21.6	13.11
50	69,804	962	20.9	13.78
51	68,842	1,001	20.2	14.54
52	67,841	1,044	19.5	15.39
53	66,797	1,091	18.8	16.33
54	65,706	1,143	18.1	17.40
55	64,563	1,199	17.4	18.57
56	63,364	1,260	16.7	19.89
57	62,104	1,325	16.1	21.34
58	60,779	1,394	15.4	22.94
59	59,385	1,468	14.7	24.72
60	57,917	1,546	14.1	26.69
61	56,371	1,628	13.5	28.88
62	54,743	1,713	12.9	31.29
63	53,030	1,800	12.3	33.94
64	51,230	1,889	11.7	36.87

5. Table of Mortality — *Continued*

Completed Age	Number Surviving at Each Age	Deaths in Each Year	Number of Years Expectation	Number Dying annually out of Each 1000
65	49,341	1,980	11.1	40.13
66	47,361	2,070	10.5	43.71
67	45,291	2,158	10.0	47.65
68	43,133	2,243	9.5	52.00
69	40,890	2,321	9.0	56.76
70	38,569	2,391	8.5	61.99
71	36,178	2,448	8.0	67.67
72	33,730	2,487	7.6	73.73
73	31,243	2,505	7.1	80.18
74	28,738	2,501	6.7	87.03
75	26,237	2,476	6.3	94.37
76	23,761	2,431	5.9	102.31
77	21,330	2,369	5.5	111.06
78	18,961	2,291	5.1	120.83
79	16,670	2,196	4.8	131.73
80	14,474	2,091	4.4	144.47
81	12,383	1,964	4.1	158.61
82	10,419	1,816	3.7	174.30
83	8,603	1,648	3.4	191.56
84	6,955	1,470	3.1	211.36
85	5,485	1,292	2.8	235.55
86	4,193	1,114	2.5	265.68
87	3,079	933	2.2	303.02
88	2,146	744	1.9	346.69
89	1,402	555	1.7	395.86
90	847	385	1.4	454.55
91	462	246	1.2	532.47
92	216	137	1.0	634.26
93	79	58	.8	734.18
94	21	18	.6	857.14
95	3	3	.5	1,000.00

6. Railway Accidents in the United States

Year ending June 30	Total	
	Killed	Injured
1897	6,437	36,731
1898	6,859	40,882
1899	7,123	44,620
1900	7,865	50,320
1901	8,455	53,339
1902	8,588	64,662
1903	9,840	76,553
1904	10,046	84,155

7. Amount of $1 at Compound Interest from One to Thirty Years

Years	3½ Per Cent	4 Per Cent	5 Per Cent	6 Per Cent
1	1.035	1.040	1.050	1.060
2	1.071	1.081	1.102	1.123
3	1.108	1.124	1.157	1.191
4	1.147	1.169	1.215	1.262
5	1.187	1.216	1.276	1.338
6	1.229	1.265	1.340	1.418
7	1.272	1.315	1.407	1.503
8	1.316	1.368	1.477	1.593
9	1.362	1.423	1.551	1.689
10	1.410	1.480	1.628	1.790
11	1.460	1.539	1.710	1.898
12	1.511	1.601	1.795	2.012
13	1.564	1.665	1.885	2.132
14	1.618	1.731	1.979	2.260
15	1.675	1.800	2.078	2.396
16	1.734	1.873	2.182	2.540
17	1.794	1.947	2.292	2.692
18	1.857	2.025	2.406	2.854
19	1.922	2.106	2.527	3.025
20	1.989	2.191	2.653	3.207
21	2.059	2.278	2.786	3.399
22	2.131	2.369	2.925	3.603
23	2.206	2.464	3.071	3.819
24	2.283	2.563	3.225	4.048
25	2.363	2.665	3.386	4.291
26	2.446	2.772	3.555	4.549
27	2.531	2.883	3.733	4.822
28	2.620	2.998	3.920	5.111
29	2.711	3.118	4.116	5.418
30	2.806	3.243	4.321	5.743

8. Amount of $1 Annually Deposited at Compound Interest

Years	$3\frac{1}{2}$ Per Cent	4 Per Cent	5 Per Cent	6 Per Cent
1	1.000	1.000	1.000	1.000
2	2.035	2.040	2.050	2.060
3	3.106	3.121	3.152	3.183
4	4.215	4.246	4.310	4.374
5	5.363	5.416	5.525	5.637
6	6.550	6.633	6.801	6.975
7	7.779	7.898	8.142	8.393
8	9.052	9.214	9.549	9.897
9	10.368	10.582	11.026	11.491
10	11.731	12.006	12.577	13.180
11	13.142	13.486	14.206	14.971
12	14.602	15.025	15.917	16.869
13	16.113	16.626	17.713	18.882
14	17.677	18.291	19.598	21.015
15	19.296	20.023	21.578	23.276
16	20.971	21.824	23.657	25.672
17	22.705	23.697	25.840	28.212
18	24.500	25.645	28.132	30.905
19	26.357	27.671	30.539	33.760
20	28.280	29.778	33.066	36.785
21	30.270	31.969	35.719	39.992
22	32.328	34.248	38.505	43.392
23	34.460	36.617	41.430	46.995
24	36.666	39.082	44.502	50.815
25	38.949	41.645	47.727	54.864
26	41.313	44.311	51.113	59.156
27	43.759	47.084	54.669	63.705
28	46.290	49.967	58.402	68.528
29	48.910	52.966	62.322	73.639

III. TABLES

TABLE 1

Squares, Cubes, Square Roots and Reciprocals of Numbers from 1 to 100

The squares, cubes, and reciprocals of decimal fractions can be obtained by shifting the decimal point. Thus $4.2^2 = 17.64$, $4.2^3 = 74.088$, $\frac{1}{4.2} = .24$. For square roots, however, this method fails, and Table 2 has to be used.

x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x
1	1	1	1.000	1.000	1
2	4	8	1.414	.500	2
3	9	27	1.732	.333	3
4	16	64	2.000	.250	4
5	25	125	2.236	.200	5
6	36	216	2.449	.167	6
7	49	343	2.646	.143	7
8	64	512	2.828	.125	8
9	81	729	3.000	.111	9
10	1 00	1 000	3.162	.100	10
11	1 21	1 331	3.317	.091	11
12	1 44	1 728	3.464	.083	12
13	1 69	2 197	3.606	.077	13
14	1 96	2 744	3.742	.071	14
15	2 25	3 375	3.873	.067	15
16	2 56	4 096	4.000	.063	16
17	2 89	4 913	4.123	.059	17
18	3 24	5 832	4.243	.056	18
19	3 61	6 859	4.359	.053	19
20	4 00	8 000	4.472	.050	20
x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x

x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x
21	4 41	9 261	4.583	.048	21
22	4 84	10 648	4.690	.045	22
23	5 29	12 167	4.796	.043	23
24	5 76	13 824	4.899	.042	24
25	6 25	15 625	5.000	.040	25
26	6 76	17 576	5.099	.039	26
27	7 29	19 683	5.196	.037	27
28	7 84	21 952	5.292	.036	28
29	8 41	24 389	5.385	.034	29
30	9 00	27 000	5.477	.033	30
31	9 61	29 791	5.568	.032	31
32	10 24	32 768	5.657	.031	32
33	10 89	35 937	5.745	.030	33
34	11 56	39 304	5.831	.029	34
35	12 25	42 875	5.916	.029	35
36	12 96	46 656	6.000	.028	36
37	13 69	50 653	6.083	.027	37
38	14 44	54 872	6.164	.026	38
39	15 21	59 319	6.245	.026	39
40	16 00	64 000	6.325	.025	40
41	16 81	68 921	6.403	.024	41
42	17 64	74 088	6.481	.024	42
43	18 49	79 507	6.557	.023	43
44	19 36	85 184	6.633	.023	44
45	20 25	91 125	6.708	.022	45
46	21 16	97 336	6.782	.022	46
47	22 09	103 823	6.856	.021	47
48	23 04	110 592	6.928	.021	48
49	24 01	117 649	7.000	.020	49
50	25 00	125 000	7.071	.020	50
51	26 01	132 651	7.141	.020	51
52	27 04	140 608	7.211	.019	52
53	28 09	148 877	7.280	.019	53
54	29 16	157 464	7.348	.019	54
55	30 25	166 375	7.416	.018	55
56	31 36	175 616	7.483	.018	56
57	32 49	185 193	7.550	.018	57
58	33 64	195 112	7.616	.017	58
59	34 81	205 379	7.681	.017	59
60	36 00	216 000	7.746	.017	60
x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x

x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x
61	37 21	226 981	7.810	.016	61
62	38 44	238 328	7.874	.016	62
63	39 69	250 047	7.937	.016	63
64	40 96	262 144	8.000	.016	64
65	42 25	274 625	8.062	.015	65
66	43 56	287 496	8.124	.015	66
67	44 89	300 763	8.185	.015	67
68	46 24	314 432	8.246	.015	68
69	47 61	338 509	8.307	.014	69
70	49 00	343 000	8.367	.014	70
71	50 41	357 911	8.426	.014	71
72	51 84	373 248	8.485	.014	72
73	53 29	389 017	8.544	.014	73
74	54 76	405 224	8.602	.014	74
75	56 25	421 875	8.660	.013	75
76	57 76	438 976	8.718	.013	76
77	59 29	456 533	8.775	.013	77
78	60 84	474 552	8.832	.013	78
79	62 41	493 039	8.888	.013	79
80	64 00	512 000	8.944	.013	80
81	65 61	531 441	9.000	.012	81
82	67 24	551 368	9.055	.012	82
83	68 89	571 787	9.110	.012	83
84	70 56	592 704	9.165	.012	84
85	72 25	614 125	9.219	.012	85
86	73 96	636 056	9.274	.012	86
87	75 69	658 503	9.327	.011	87
88	77 44	681 472	9.381	.011	88
89	79 21	704 969	9.434	.011	89
90	81 00	729 000	9.487	.011	90
91	82 81	753 571	9.539	.011	91
92	84 64	778 688	9.592	.011	92
93	86 49	804 357	9.644	.011	93
94	88 36	830 584	9.695	.011	94
95	90 25	857 375	9.747	.011	95
96	92 16	884 736	9.798	.010	96
97	94 09	912 673	9.849	.010	97
98	96 04	941 192	9.899	.010	98
99	98 01	970 299	9.950	.010	99
100	100 00	1 000 000	10.000	.010	100
x	x^2	x^3	$\sqrt{x}$	$\frac{1}{x}$	x

TABLE 2

Square Roots of Numbers from 1 to 9.9

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.000	0.316	0.447	0.548	0.632	0.707	0.775	0.837	0.894	0.949
1	1.000	1.049	1.095	1.140	1.183	1.225	1.265	1.304	1.342	1.378
2	1.414	1.449	1.483	1.517	1.549	1.581	1.612	1.643	1.673	1.703
3	1.732	1.761	1.789	1.817	1.844	1.871	1.897	1.924	1.949	1.975
4	2.000	2.025	2.049	2.074	2.098	2.121	2.145	2.168	2.191	2.214
5	2.236	2.258	2.280	2.302	2.324	2.345	2.366	2.387	2.408	2.429
6	2.449	2.470	2.490	2.510	2.530	2.550	2.569	2.588	2.608	2.627
7	2.646	2.665	2.683	2.702	2.720	2.739	2.757	2.775	2.793	2.811
8	2.828	2.846	2.864	2.881	2.898	2.915	2.933	2.950	2.966	2.983
9	3.000	3.017	3.033	3.050	3.066	3.082	3.098	3.114	3.130	3.146

ANSWERS TO EXERCISES

Exercise 1. Pages 2, 3

5. 5.66. **6.** 5. **7.** In a line $\parallel XX'$ passing through (0, 4).
8. In the y-axis. **9.** In the x-axis.
10. The line $\parallel XX'$ passing through (0, 3).
11. The ordinate. **12.** (0, 0).

Exercise 2. Pages 6, 7, 8

1. (*a*) 6°, 5.9°, 5.25°, 2.5°. (*b*) 1 : 40 P.M. and 5 P.M; 1 P.M. and 6 P.M.; 11 A.M. and 8.40 P.M.; 10 P.M.; 9.20 P.M.
(*c*) 3.15 P.M. (*d*) 7 +°. (*e*) 1 P.M. to 6 P.M.
(*f*) 12 M. to 12.30 P.M.; 6.40 P.M. to 7.20 P.M.
(*g*) 11 A.M. to 9.20 P.M. (*h*) 9.20 P.M. on. (*i*) 4.75°. (*k*) 6 P.M.
(*l*) 11 A.M. to 3.15 P.M. (*m*) 3.15 P.M. on.
(*n*) Between 3 P.M. and 4 P.M.
(*o*) Between 12 M. and 1 P.M.; and between 11 A.M. and 12 M.
2. (*a*) San Francisco. (*b*) Bismarck. (*c*) April 20 and Sept. 20.
(*d*) During April. (*e*) 25°.
3. (*c*) 18° C. (*d*) 8.1 grams. (*e*) 15 grams.

Exercise 3. Pages 10, 11

25. (*a*) 12.25. (*b*) 2.25. (*c*) 7.84. (*d*) 3.61.
(*e*) 2.5. (*f*) 3.5. (*g*) 2.24. (*h*) .55.
26. (*a*) 4.25, – 1.75, – 1.75. (*b*) 2; 3.73 and .27; 3.87 and .13.
(*c*) – 2. (*d*) 2. (*e*) 3.41 and .59. (*f*) 3.41 and .59.
(*g*) 3 and 1. (*h*) 0 and 4.
27. (*a*) 2.75, – 3.25, 1.5. (*b*) 3.24, – 1.24. (*c*) 3. (*d*) 1.
(*e*) 2.73, – .73. (*f*) 2.73, – .73. (*g*) 2.4, – .4. (*h*) 2.4, – .4.
28. (*b*) 31.25 meters. (*c*) 2.24 seconds.

Exercise 4. Page 12

7. (*b*) $-18\frac{1}{3}$° C., $-12\frac{7}{9}$° C., -10° C., 0° C. (*c*) 14° F., 32° F., $33\frac{4}{5}$° F.

Exercise 5. Pages 15, 16

1. 1.75. **2.** -2.5. **3.** 6. **4.** 2.67. **5.** $3, -2$. **6.** $2.79, -1.79$. **7.** $3.83, -1.83$. **8.** 3, 3. **9.** $.7, -5.7$. **10.** $5.54, -.54$. **11.** $4.37, -1.37$. **12.** $-2.16, 4.16$. **13.** $-1.93, 2.93$. **14.** $-1.92, 3.92$. **15.** $-5.62, .62$. **16.** $-1.31, 3.31$. **17.** $-1.53, -.35, 1.88$. **18.** -4.05, 2 imag. **19.** $-2.11, .25, 1.86$. **20.** 1.21, 2 imag. **21.** 1.78, 2 imag. **22.** $-1.94, .55, 1.39$. **23.** $-3.1, 3.5, 4.6$. **24.** $-.39, 5.44, 7.95$. **25.** $\pm .94, \pm 3.02$. **26.** $-2.5, 1.73$, 2 imag. **27.** $-.97, .85, 2.15, 3.97$. **28.** $-2.99, -1.15, .21, 1.9, 3.05$. **29.** 1.38.

30. (*a*) $-4.51, -1.75, 1.26$. (*b*) $-4.12, -2.4, 1.52$. (*c*) $-4.78, -1.14, .92$. (*d*) -5.19, 2 imag. (*e*) 3. (*f*) -10 to 8.5^+. (*g*) -10 or 8.5^+. (*h*) 8.5. (*i*) -3.33.

31. (*a*) $-2.84, .44, 2.4$. (*b*) $-2.65, 0, 2.65$. (*c*) $-3, 1, 2$. (*d*) $-1.68, -1.38, 3.05$. (*e*) -3.49, 2 imag. (*f*) 1, 3, 3, 1. (*g*) 2, 2, 0.

Exercise 7. Page 22

1. $x = 2.8, y = -.1$. **2.** $x = 2, y = 1$. **3.** $x = -.8, y = -2.8$. **4.** $x = 2.5, y = .7$. **5.** $x = 1.5, y = .5$. **8.** Parallel. **9.** $x = 3.6, y = -1.6$. $x = -1.6, y = 3.6$. **10.** $x = 3, y = 2$. $x = 2, y = 3$. **11.** $x = 4, y = 3$. $x = -3, y = -4$. **12.** $x = 4, y = 2$. $x = -2, y = -4$.

13. $x = 4.3, y = 1.4$. $x = -1.8, y = -3.4$. **14.** $x = 2.3, y = 1.15$. $x = -2.3, y = -1.15$. **15.** $x = \pm 4.8, y = \pm 1.3$. $x = \pm 1.3, y = \pm 4.8$. **16.** $x = \pm 3, y = \pm 1$. $x = \pm \infty, y = \mp \infty$.

Exercise 8. Page 24

1. $x = 1.82, y = -.82$. $x = -.82, y = 1.82$. **2.** $x = 4, y = 0$. $x = 0, y = -4$. **3.** $x = -7, y = -1$. $x = 1, y = 7$. **4.** $x = 2.96, y = .48$. $x = -2.16, y = -2.08$. **5.** $x = -3.4, y = -2.1$. $x = .63, y = 3.95$. **6.** $x = 2, y = 4$. $x = 0, y = 0$. **7.** $x = 3, y = 0$. $x = 1, y = -2$. **8.** $x = 1, y = \pm 3$. **9.** $x = .22, y = 1.72$. $x = -1.72, y = -.22$. **10.** $x = 0, y = \pm 1$.

Exercise 9. Page 28

1. $3, -2.$
2. $1, -2.$
3. $6, -3.$
4. $2, -5.$
5. $4, -2.$
6. $1.24, -3.24.$
7. $7.1, -2.1.$
8. $.95, 5.3.$
9. $-6, -.67.$
10. $1.8, -2.8.$
11. $4.2, -2.2.$
12. $2.7, -5.1.$
13. $2.1, -4.6.$
14. $1.5, -.7.$

Exercise 10. Pages 31, 32

1. $-60, 75.$
2. $-50, 60.$
3. $-60, -20.$
4. $-60, 20.$
5. $-80, 50.$
6. $-70, -10.$
7. $-20, 30.$
8. $-16, 8.$
9. $-30, 60.$
10. $-55, 22.$
11. $-28.3, 26.8.$
12. $-33.33, 30.$
13. $-35, 6.$
14. $-72.2, 5.5.$
15. $-.225, .525.$
16. $-.136, .736.$
17. $-1.425, .175.$
18. $-.311, .161.$
19. $-.2, .3.$
20. $-.6, .2.$
21. $-.2, .1.$
22. $-.25, .5.$

Exercise 11. Pages 34, 35

1. $13, 6, -2.75, -2.75.$
2. $-2.25, -3, -3, 3.$
3. $2.91, -2.25, -17.25.$
4. $31.25, 130.96, 47.24.$
5. $4.87, -2.87.$
6. $-9.47, -.53.$
7. -1 when $x = 2.$
8. -30 when $x = -5.$
9. -15.25 when $x = -3.5.$
10. -4.25 when $x = 2.5.$
11. $7, 130.5, 147.$

Exercise 12. Page 39

1. $5, 5.$
2. $5 \pm 2i.$
3. $-2 \pm 2i.$
4. $-4 \pm 2i.$
5. $4 \pm 3i.$
6. $5 \pm 2i.$
7. $-3.5 \pm 2.96i.$
8. $2.5 \pm 2.96i.$
9. $-1.5 \pm 4.97i.$
10. $-4.5 \pm 3.97i.$
11. $-.5 \pm .87i.$
12. $-1, -1.$
13. $-.5 \pm 1.12i.$
14. $1.5 \pm 2i.$

Exercise 13. Page 41

1. $5, -3.$
2. $3, -2.$
3. $5, 1.$
4. $-4, -1.$
5. $1 \pm 3i.$
6. $-1.27, -4.73.$

Exercise 14. Page 45

1. 2.
2. 3.
3. $-2.$
4. $-3.$
5. $1, 2, -3.$
6. $-2\frac{1}{2}, -1, 3\frac{1}{2}.$
7. $-3.3.$
8. 3.2.
9. 2.7.
10. 6.6.
11. $-2.6.$
12. 3.3.
13. 4.5
14. $-4.6.$
15. $-4.5.$
16. $-.6, 5.7, -5.2.$
17. 1.1.
18. 8.
19. $4.5, 4.5, -9.$
20. $-11.$
21. $-5, -5, 10.$
22. $-11.9.$
23. 4.1.
24. $-16.5.$

Exercise 15. Page 48

1. $-55, -41.6, -33.7, 40.9$.
2. $-18.5, -17.4, 8, 1.4$.
3. $-24, -11, -20.6, 6.9$.
4. $-454, -82, 30, 326$.
5. 3.2.
6. $2.4, .4, -2.1$.
7. -3.6.
8. 3.4.
9. -16.
10. 14.
13. $101, 73.7, -.2$.
14. $-133, -445, -67$.
15. -31.

Exercise 16. Page 52

1. $-1, -1, 2$.
2. $-2, 1, 1$.
3. $-2, 1 \pm 2\,i$.
4. $2, -1 \pm .5\,i$.
5. $2, -1 \pm 1.5\,i$.
6. $-2, 1 \pm 3.46\,i$.
7. $-4, 2 \pm 1.73\,i$.
8. $-7, 3.5 \pm 5.27\,i$.
9. $-1.5, .75 \pm .43\,i$.
10. $2, -1 \pm .87\,i$.
11. $1, -.5 \pm 2.18\,i$.
12. $-1.46, .73 \pm 1.9\,i$.

Exercise 17. Pages 56, 57

1. $-1, 1, 3$.
2. $1, 3, 5$.
3. $-1, 2, 5$.
4. $1, 2, 5$.
5. $-1, -2, -4$.
6. $-2, 1, 3$.
7. $-1.52, .43, 3.09$.
8. -1.18, 2 imag.
9. $-4.19, -1, 1.19$.
10. -1.88, 2 imag.
11. $-3.61, -.87, .48$.
12. $-2, .6, 2$.
13. $-1.9, .76, 3.14$.
14. $-1, .5, 3$.
15. $-1.5, .5, 4$.
16. $-2, .5, 4.5$.
17. -9.28, 2 imag.
18. $-1, 2 \pm i$.
19. $-4, -1 \pm i$.
20. $-1, 2 \pm 1.41\,i$.
21. $-4, -1 \pm 2\,i$.
22. $4, 1 \pm 1.41\,i$.
23. $-3, 1 \pm 1.73\,i$.
24. $-3.84, 1.42 \pm 1.37\,i$.
25. $1, 4 \pm 2\,i$.

Exercise 18. Page 58

1. (*a*) 5.26. (*b*) 1.85. (*c*) 1.

Exercise 19. Page 61

1. $-2, 1.66$, 2 imag.
2. $-4, -1, 2, 3$.
3. $-2, 1$, 2 imag.
4. $-4, -1.83, 2, 3.83$.
5. $-2, -1, 1, 2$.
6. $1, 3$, 2 imag.
7. $-2.69, -.63$, 2 imag.
8. $1, 2.76$, 2 imag.
9. $-2.73, -1, .73, 3$.
10. $-2.22, 2.54$, 2 imag.
11. $-2.62, -.38, 1, 2$.
12. $-3, -2, 1, 4$.

Exercise 20. Pages 62, 63

1. $-6, -2, 1, 7$.
2. $-5, -3, 1, 7$.
3. $-4, -2, 1, 5$.
4. $-5, -3, 2, 6$.
5. $-8, -2, 1, 9$.
6. $-8, -2, 3, 7$.
7. $-6, -4, 3, 7$.
8. $-7, -3, 3, 7$.
9. $-7, -2, 3, 6$.
10. $-6, -3, 2, 7$.

Exercise 21. Page 65

1. $-3, -1, 2 \pm i$.
2. $-3, -1, 2 \pm 1.41\, i$.
3. $-1, 3, -1 \pm 1.41\, i$.
4. $-1, 3, -1 \pm i$.
5. $1.47, 2, -1.73 \pm 1.04\, i$.
6. $-1.82, 1, .41 \pm 2.31\, i$.
7. $1, 3.61, -2.31 \pm 1.15\, i$.
8. $-4, 2, 1 \pm i$.
9. $-3, 1, 1 \pm i$.
10. $-3, -1, 2 \pm i$.

Exercise 22. Page 68

1. 15.
2. 69.
3. 227.
4. 89.94.
5. 45.07.

Exercise 23. Page 71

1. $-5, -2, 1, 2$.
2. $-3, -1, 2, 6$.
3. $-3, 1, 4, 6$.
4. $-5, -3, -1, 1$.
5. $-4, -2, 0, 2$.
6. $-3, -1, 1, 5$.
7. $-5, -3, 1, 4$.
8. $-7, -4, 1, 2$.
9. $1, 2, 3, 4$.
10. $-4, -2, 1, 2$.
11. $-2.41, .41, 2, 4$.

Exercise 24. Pages 75, 76, 77

1. 6 hrs.; 18 miles.
2. 18 miles.
3. $4\frac{1}{2}$ hrs.
4. $3\frac{2}{3}$ hrs.
5. $2\frac{1}{2}$ hrs.; $7\frac{1}{2}$ miles.
6. 2 : 50.
7. (*a*) 3. (*b*) 3. (*c*) 4. (*d*) 4.
8. 4.
9. 2 : 10.9.
10. 3 : 49.1.
11. 6 : 16.4, 6 : 49.1.
12. 2 hrs.
13. 1.28 hrs.
14. (*a*) 72.2 meters.
    (*b*) 155 meters.

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